Lecture Notes

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September 20, 2020

Contents

1 Graph Theory

To quote Bud Brown, “Graph theory is a subject whose deceptive simplicity masks its vast applicability.” Graph theory provides simple mathematical structures known as graphs to model the relations of various objects. The applications are numerous, including efficient storage of chemicals (graph coloring), optimal assignments (matchings), distribution networks (flows), efficient storage of data (tree-based data structures), and machine learning. In automata theory, we use directed graphs to provide a visual representation of our machines. Many elementary notions from graph theory, such as path-finding and walks, come up as a result. In complexity theory, many combinatorial optimization problems of interest are graph theoretic in nature. Therefore, it is important to discuss basic notions from graph theory. We begin with the basic definition of a graph.

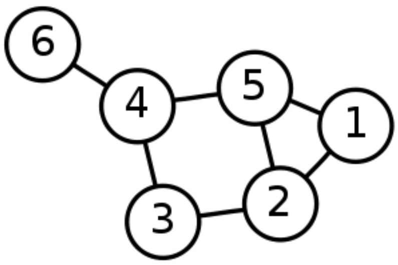
The exposition in this section has benefitted from [1, 2, 3].

1.1 Introduction to Graphs

Definition 1 (Simple Graph)**.** A simple graph is a two-tuple $G(V, E)$ where V is a set of vertices and $E \subset \binom{V}{2}$.

By convention, a simple graph is referred to as a *graph*, and an edge $\{i, j\}$ is written as ij. In simple graphs, $ij = ji$. Two vertices i, j are said to be *adjacent* if $ij \in E(G)$. Now let's consider an example of a graph.

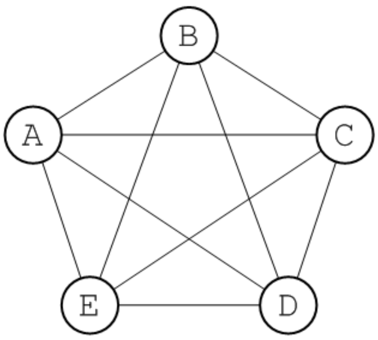
Example 1. Let $G(V, E)$ be the graph where $V = \{1, 2, \ldots, 6\}$ and $E = \{12, 15, 23, 25, 34, 45, 46\}$. This graph is pictured below.



We now introduce several common classes of graphs.

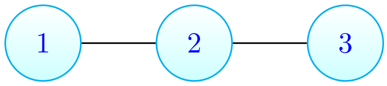
Definition 2 (Complete Graph)**.** The complete graph, denoted K_n, has the vertex set $V = \{1, 2, \ldots, n\}$ and edge set E which consists of **all** two-element subsets of V. That is, K_n has all possible edges between vertices.

Example 2. The complete graph on five vertices K_5 is pictured below.



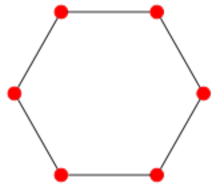
Definition 3 (Path Graph)**.** The path graph, dentoed P_n, has vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{\{i, i + 1\} : 1 \leq i \leq n - 1\}$.

Example 3. The path on three vertices P_3 is shown below.



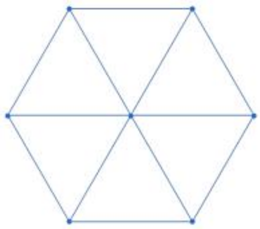
Definition 4 (Cycle Graph)**.** Let $n \geq 3$. The cycle graph, denoted C_n, has the vertex set $V = \{1, 2, \dots, n\}$ and the edge set $E = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$.

Example 4. Intuitively, C_n can be thought of as the regular n-gon. So C_3 is a triangle, C_4 is a quadrilateral, and C_5 is a pentagon. The graph C_6 is pictured below.



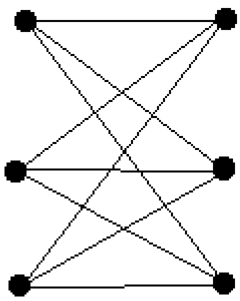
Definition 5 (Wheel Graph)**.** Let $n \geq 4$. The wheel graph, denoted W_n, is constructed by joining a vertex n to each vertex of C_{n-1}. So we take $C_{n-1} \dot{\cup} n$ and add the edges vn for each $v \in [n-1]$.

Example 5. The wheel graph on seven vertices W_7 is pictured below.



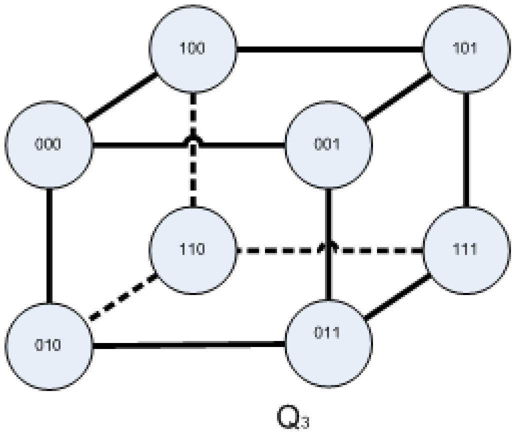
Definition 6 (Bipartite Graph)**.** A bipartite graph $G(V, E)$ has a vertex set $V = X \dot{\cup} Y$, with edge set $E \subset \{xy : x \in X, y \in Y\}$. That is, no two vertices in the same part of V are adjacent. So no two vertices in X are adjacent, and no two vertices in Y are adjacent.

Example 6. A common class of bipartite graphs include even-cycles C_{2n}. The complete bipartite graph is another common example. We denote the complete bipartite graph as $K_{m,n}$ which has vertex partitions $X \dot{\cup} Y$ where $|X| = m$ and $|Y| = n$. The edge set $E(K_{m,n}) = \{xy : x \in X, y \in Y\}$. The graph $K_{3,3}$ is pictured below.



Definition 7 (Hypercube)**.** The hypercube, denoted Q_n, has vertex set $V = \{0, 1\}^n$. Two vertices are adjacent if the binary strings differ in precisely one component.

Example 7. The hypercube Q_2 is isomorphic to C_4 (isomorphism roughly means that two graphs are the same, which we will formally define later). The hypercube Q_3 is pictured below.



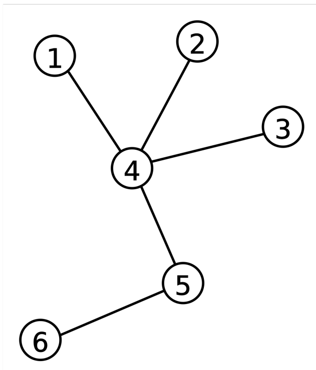
Definition 8 (Connected Graph)**.** A graph $G(V, E)$ is said to be connected if for every $u, v \in V(G)$, there exists a $u - v$ path in G. A graph is said to be *disconnected* if it is not connected; and each connected subgraph is known as a *component*.

Example 8. So far, every graph presented has been connected. If we take two disjoint copies of any of the above graphs, their union forms a disconnected graph.

Definition 9 (Tree)**.** A Tree is a connected, acyclic graph.

Example 9. A path is an example of a tree. Additional examples include the binary search tree, the binary heap, and spanning trees of graphs.

Example 10. The following is an example of a tree.



Definition 10 (Degree)**.** Let $G(V, E)$ be a graph and let $v \in V(G)$. The degree of v, dentoed $\deg(v)$ is the number of edges containing v. That is, $\deg(v) = |\{vx : vx \in E(G)\}|$.

Example 11. Each vertex in the Cycle graph C_n has degree 2. In Example 1, $\deg(6) = 1$ and $\deg(5) = 3$.

Theorem 1.1 (Handshake Lemma)**.** *Let $G(V, E)$ be a graph. We have:*

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|.$$

Proof. The proof is by double counting. The term $\deg(v)$ counts the number of edges incident to v. Each edge has two endpoints v and x, for some other $x \in V(G)$. So the edge vx is double counted in both $\deg(v)$ and $\deg(x)$. Thus,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

□

Remark: The Handshake Lemma is a *necessary condition* for a graph to exist. That is, all graphs satisfy the Handshake Lemma. Consider the following: does there exist a graph on 11 vertices each having degree

5? By the Handshake Lemma, $11 \cdot 5 = 2|E(G)|$. However, 55 is not even, so no such graph exists. Note that the Handshake Lemma is not a *sufficient condition*. That is, there exist degree sequences such as $(3, 3, 1, 1)$ satisfying the Handshake Lemma which are not realizable by any graph. Theorems such as Havel-Hakimi and Erdós-Gallai provide conditions that are both sufficient and necessary for a degree sequence to be realizable by some graph.

Next, the notion of a walk will be introduced.

Definition 11 (Walk)**.** Let $G(V, E)$ be a graph. A walk of length n is a sequence $(v_i)_{i=0}^{n}$ such that $v_i v_{i+1} \in E(G)$ for all $i \in \{0, \ldots, n-1\}$. If $v_0 = v_n$, the walk is said to be *closed.*

Let us develop some intuition for a walk. We start a given vertex v_0. Then we visit one of v_0's neighbors, which we call v_1. Next, we visit one of v_1's neighbors, which we call v_2. We continue this construction for the desired length of the walk. The key difference between a walk and a path is that a walk can repeat vertices, while all vertices in a path are distinct.

Example 12. Consider a walk on the hypercube Q_3. The sequence of vertices $(000, 100, 110, 111, 101)$ forms a walk, while $(000, 100, 110, 111, 101, 001, 000)$ is a closed walk. The sequence $(000, 111)$ is not a walk because 000 and 111 are not adjacent in Q_3.

We now define the adjacency matrix, which is useful for enumerating walks of a given length.

Definition 12 (Adjacency Matrix)**.** Let $G(V, E)$ be a graph. The adjacency matrix A is an $n \times n$ matrix where:

$$A_{ij} = \begin{cases} 1 & : ij \in E(G) \\ 0 & : ij \notin E(G) \end{cases} \tag{1}$$

Example 13. Consider the adjacency matrix for the graph K_5:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{2}$$

Theorem 1.2. *Let $G(V, E)$ be a graph, and let A be its adjacency matrix. For each $n \in \mathbb{Z}^+$, A_{ij}^n counts the number of walks of length n starting at vertex i and ending at vertex j.*

Proof. The proof is by induction on n. When $n = 1$, we have A. By definition $A_{ij} = 1$ iff $ij \in E(G)$. All walks of length 1 correspond to the edges incident to i, so the theorem holds true when $n = 1$. Now fix $k \geq 1$ and suppose that for each $m \in [k]$ that A_{ij}^m counts the number of $i - j$ walks of length m. The $k + 1$ case will now be shown.

Consider $A^{k+1} = A^k \cdot A$ by associativity. By the inductive hypothesis, A_{ij}^k and A_{ij} count the number of $i - j$ walks of length k and 1 respectively. Observe that:

$$A_{ij}^{k+1} = \sum_{x=1}^{n} A_{ix}^k A_{xj}$$

So A_{ix}^k counts the number of ix walks of length k, and $A_{xj} = 1$ iff $xj \in E(G)$. Adding the edge xj to an $i - x$ walk of length k forms an $i - j$ walk of length $k + 1$. The result follows by induction. $\square$

We will prove one more theorem before concluding with the graph theory section. In order to prove this theorem, the following lemma (or helper theorem) is needed.

Lemma 1.1. *Let $G(V, E)$ be a graph. Every closed walk of odd length at least 3 in G contains an odd-cycle.*

Proof. The proof is by induction on the length of the walk. Note that a closed walk of length 3 forms a K_3. Now fix $k \geq 1$ and suppose the that any closed walk of odd length up to $2k+1$ has an odd-cycle. We prove true for walks of length $2k+3$. Let $(v_i)_{i=0}^{2k+3}$ be a walk closed of odd length. If $v_0 = v_{2k+3}$ are the only repeated vertices, then the walk itself is an odd cycle and we are done. Otherwise, suppose $v_i = v_j$ for some $0 \leq i < j \leq 2k+3$. If the walk $(v_t)_{t=i}^{k}$ is odd, then there exists an odd cycle by the inductive hypothesis. Otherwise, the walk $W = (v_0, \ldots, v_i, v_{j+1}, \ldots, v_{2k+3})$ is of odd length at most $2k+1$. So by the inductive hypothesis, W has an odd cycle. So the lemma holds by induction. □

We now characterize bipartite graphs.

Theorem 1.3. *A graph $G(V, E)$ is bipartite if and only if it contains no cycles of odd length.*

Proof. Suppose first that G is bipartite with parts X and Y. Now consider a walk of length n. As no vertices in a fixed part are adjacent, only walks of even lengths can end back in the same part as the staring vertex. A cycle is a walk where all vertices are distinct, save for v_0 and v_n which are the same. Therefore, no cycle of odd length exists in G.

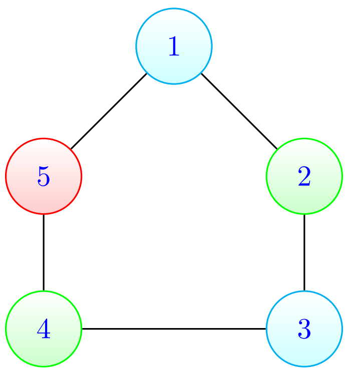
Conversely, suppose G has no cycles of odd length. We construct a bipartition of $V(G)$. Without loss of generality, suppose G is connected. For if G is not connected, we apply the same construction to each connected component. Fix the vertex v. Let $X = \{u \in V(G) : d(u,v) \text{ is even }\}$, where $d(u,v)$ denotes the distance or length of the shortest uv path. Let $Y = \{u \in V(G) : d(u,v) \text{ is odd }\}$. Clearly, $X \cap Y = \emptyset$. So it suffices to show no vertices within X are adjacent, and no vertices within Y are adjacent. Fix $v \in X$ and suppose to the contrary that two vertices in $y_1, y_2 \in Y$ are adjacent. Then there exists a closed walk of odd length $(v, \ldots, y_1, y_2, \ldots v)$. By Lemma 1.1, G must contain an odd-cycle, a contradiction. By similar argument, no vertices in X can be adjacent. So G is bipartite with bipartition $X \dot{\cup} Y$. □

1.2 Coloring

In this section, we introduce the notion of a graph vertex coloring. Informally, a proper vertex coloring of a graph $G(V, E)$ is an assignment of colors to V such that if two vertices $u, v \in V(G)$ are adjacent, then u and v receive different colors. This is formalized as follows.

Definition 13. A *vertex coloring* of a graph $G(V, E)$ is a function $\phi : V(G) \to \{1, 2, \ldots, n\}$ such that whenever $uv \in E(G)$, $\phi(u) \neq \phi(v)$. The *chromatic number* of G, denoted $\chi(G)$, is the smallest $n \in \mathbb{N}$ such that there exists a coloring $\phi : V(G) \to \{1, 2, \ldots, n\}$.

Example 14. Consider the graph C_5, shown below. We show that $\chi(C_5) = 3$. First, we provide a coloring of C_5 to show that $\chi(C_5) \leq 3$. Consider the following.



Now observe that vertex 5 **cannot** be colored using either green or blue. In particular, regardless of how we color the vertices of C_5 using green and blue, there will always be a vertex v adjacent to both a green and blue vertex. For this reason, a third color for v. So $\chi(C_5) \geq 3$, and we may conclude that $\chi(C_5) = 3$. More generally, if n is odd, $\chi(C_n) = 3$, by the same argument as for C_5.

Example 15. Suppose n is even, and consider the cycle graph C_n. As C_n has an edge, $\chi(C_n) \geq 2$. To show that $\chi(C_n) \leq 2$, we exhibit a two-coloring of C_n. Here, we assign the even indexed vertices of C_n the color `green`, and we assign the odd indexed vertices of C_n `blue`. As the number of vertices in C_n is even, we have that odd-indexed vertices will only be adjacent to even-indexed vertices, and vice-versa. In particular, vertex n is even, so we will not have a case such as in C_5, where two odd vertices are adjacent. For this reason, the two-coloring we provided is valid. Thus, $\chi(C_n) \leq 2$, and we conclude that $\chi(C_n) = 2$.

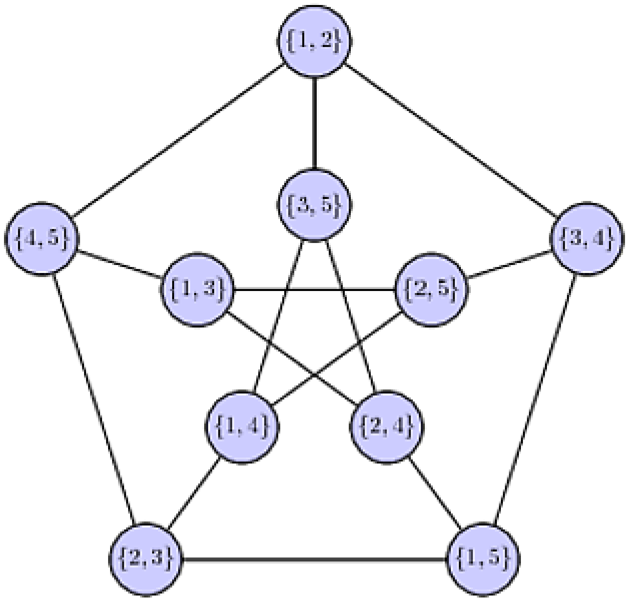
Example 16. Consider the complete graph on n vertices, K_n. We claim that $\chi(K_n) = n$.

- First, we establish the upper bound that $\chi(K_n) \leq n$. Consider the coloring φ, which assigns vertex i the color i. As each vertex receives a distinct color, no two adjacent vertices receive the same color. So φ is a valid coloring. Thus, $\chi(K_n) \leq n$.
- We next establish that $\chi(K_n) > n - 1$. To see this, observe that for distinct any $i, j \in \{1, 2, \ldots, n\}$, vertices i and j are adjacent in K_n. So we cannot assign i and j the same color. It follows that we must use at least n colors to properly color K_n. So $\chi(K_n) \geq n$.

We conclude that $\chi(K_n) = n$, as desired.

1.2.1 Exercises

(Required) Problem 1. Let $\binom{[5]}{2}$ denote the set of 2-element subsets of [5]. The *Petersen graph* is a simple graph $\mathcal{P}$ with vertex set $\binom{[5]}{2}$. Two vertices $\{a, b\}$ and $\{c, d\}$ are adjacent in the Petersen graph if and only if $\{a, b\} \cap \{c, d\} = \emptyset$. Determine $\chi(\mathcal{P})$.



(Required) Problem 2. Let $G(V, E)$ be a graph. The *clique number* of G, dentoed $\omega(G)$, is the order of the largest complete subgraph of G. That is, if $\ell = \omega(G)$, then K_ℓ is a subgraph of G; and for any $h > \ell$, K_h is **not** a subgraph of G. Explain why $\omega(G) \leq \chi(G)$.

(Required) Problem 3. Let $G(V, E)$ be a graph.

(a) Prove that if G is bipartite, then G can be colored using at most 2 colors.

(b) Conversely, suppose that G can be colored using at most 2 colors. Prove that G is bipartite.

(Required) Problem 4. Let T be a tree with at least 2 vertices. Determine $\chi(T)$.

(Required) Problem 5. Let $d > 0$. Determine $\chi(Q_d)$.

(Required) Problem 6. For this problem, we consider the Wheel graph W_n, for $n \geq 4$.

(a) Determine $\chi(W_n)$.

(b) Determine all values of n such that W_n is bipartite. Clearly justify your answer.

References

[1] Reinhard Diestel. *Graph Theory*. 2010.

[2] Michael Levet. Theory of computation- lecture notes. 2018. URL `http://people.math.sc.edu/mlevet/Lecture_Notes.pdf`.

[3] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2 edition, September 2000. ISBN 0130144002.