The Game of Nim

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We consider the following two-person game, called *Nim*, in which players alternate turns. The game begins with a pile of n (identical) stones. During a player's turn, they may remove either $1, 2$, or 3 stones from the pile. If a player cannot make a move, that player loses. This game is denoted as a $(1, 2, 3)$-Nim game, in light of the allowed moves of removing $1, 2$, or 3 stones. When we refer to *Nim*, we will restrict attention to $(1, 2, 3)$-Nim.

Example 1. We work through an example of the Game of Nim. Suppose we start with $n = 5$ stones. One possible outcome is as follows:

- Player 1 selects 3 stones, leaving Player 2 with 2 stones as their turn starts.
- Player 2 selects both remaining stones, winning the game.

This is, of course, not the only possible outcome. There is, in fact, an outcome where Player 1 wins. Consider the following gameplay, which we label **Gameplay 2**.

- Player 1 selects only 1 stone, leaving Player 2 with 4 stones as their turn starts.
- Player 2 selects 3 stones, leaving Player 1 with 1 stone as their turn starts.
- Player 1 selects the last stone, winning the game.

(Recommended) Problem 1. Does the outcome of Gameplay 2 change if we modify Player 2's strategy?

(Recommended) Problem 2. To develop better intuition for Nim, play with each other using the following number of initial stones:

- $n = 3$ stones.
- $n = 4$ stones.
- $n = 5$ stones.
- $n = 6$ stones.
- $n = 7$ stones.
- $n = 8$ stones.

For what values of n that you considered did Player 1 have a winning strategy? What about Player 2? Comment on your observations. <u>Assume that both players are intending to win the game.</u>

(Recommended) Problem 3. Does Nim exhibit optimal substructure? That is, can we use what we know about whether Player 1 has a winning strategy for a game starting with k stones to determine whether Player 1 has a winning strategy for $n > k$ stones? Precisely, if the game starts with n stones, what sub-problems (in terms of n) does Player 1 need to consider to determine if they have a winning strategy?

[**Hint:** It may be helpful to start with $n = 5$, $n = 6$, and $n = 7$ as examples. If the game starts with 5 stones, which sub-problems does Player 1 need to consider to determine if they have a winning strategy?]

(Recommended) Problem 4. Using the bottom-up dynamic programming technique, construct a lookup table indicating whether Player 1 has a winning strategy when started with n stones. Fill in the lookup table for $n \in \{1, \ldots, 8\}$.

(Recommended) Problem 5. Look closely at your lookup table. Is there a pattern that stands out for when Player 1 has a winning strategy? Your goal is to provide a condition for the statement: When the game is started with n stones, Player 1 has a winning strategy if and only if $\langle$some condition about $n\rangle$.