CSCI 3104 Recitation- Graph Theory

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1 Learning Objectives

The primary goal of recitation this week is to (re)familiarize students with basic notions of graph theory that arise in this course, as well as future CS courses. In particular, we have the following learning outcomes.

- Students will correctly identify basic families of graphs including paths, cycles, wheels, complete graphs, hypercubes, bipartite graphs, and trees.
- Students will determine whether certain graphs are bipartite.
- Students will determine the number of edges in a graph using counting arguments. (These will be especially important in our discussions of P vs. NP.)

2 Graph Theory

(Recommended) Problem 1. Determine whether each of the following graphs are bipartite. Justify your answer. You may use, without proof, the fact that a graph G is bipartite if and only if G does not have any cycles of odd length.

(a) A *tree* is a graph $T(V, E)$ that is connected and acyclic (that is, T does not contain any cycles).

(b) For which values of $n \geq 3$ is the cycle graph C_n bipartite.

(c) For which values of $n \geq 4$ is the wheel graph W_n bipartite.

(d) For which values of $n \geq 1$ is the complete graph K_n bipartite.

(Recommended) Problem 2. Do the following.

(a) For which values of $n \geq 3$ is the cycle graph C_n also a complete graph.

(b) For which values of $n \geq 4$ is the wheel graph W_n also a complete graph.

(c) For which values of $d \geq 0$ is the hypercube Q_d also a complete graph.

(Recommended) Problem 3. Recall that the graph Q_d is the hypercube, whose vertex set is the set of strings $\{0,1\}^d$ (i.e., the set of binary strings of length n) and two vertices v_1, v_2 are adjacent if and only if they differ in precisely one position. Answer the following:

(a) How many vertices belong to Q_d?

(b) How many edges belong to Q_d?

(c) Suppose that a string $\omega \in \{0,1\}^d$ has exactly i digits that are 1's. Explain why ω's neighbors in Q_d only have either exactly $(i+1)$ or exactly $(i-1)$ digits that are 1's.

(d) Prove that the graph Q_d is bipartite by giving an explicit bipartition of $\{0,1\}^d$. [**Hint:** Use part (c) to construct your bipartition.]

(Recommended) Problem 4. Do the following.

(a) Let $d \geq 1$. Show how to construct Q_d using two copies of Q_{d-1}.

(b) A *Hamiltonian cycle* in a graph G is a cycle graph that includes every vertex of G. Prove by induction that Q_d has a Hamiltonian cycle for all $d \geq 2$. [**Hint:** Use part (a) in the inductive step.]

(Recommended) Problem 5. Let $G(V, E)$ be a graph, and let $u, v \in V(G)$. Suppose that G has a $u - v$ path. Explain why G has a $u - v$ walk.

(Recommended) Problem 6. There used to be 26 teams in the NFL, with 13 teams in each of two conferences. The NFL had a rule that said that each team's 14 game schedule needed to include 11 games against teams in its own conference and 3 games against teams in the other conference. By modeling this problem as a graph, determine that this rule could not be satisfied.

(Recommended) Problem 7. Explain by $1 + 2 + \ldots + n = \binom{n+1}{2}$ by counting handshakes in two ways. [**Hint:** What familiar graph has $\binom{n+1}{2}$ edges?]

(Recommended) Problem 8. Suppose we have n couples at a party. Each person shakes hands with everyone else, with the exception of their partner. Note that a person does not shake hands with themselves. How many handshakes occurred at this party?