Graph Theory: Shannon Capacity Optional Problem Set

May The Force Be With:

Instructions: This problem set is **optional.** Individual parts are listed as Required, Advanced, Challenge, or Open to indicate difficulty. Open problems are unsolved problems in mathematics. Successfully solving an open problem will result in a publication, fame, and eternal glory. You may use a previous part to answer subsequent questions, even if you do not answer that earlier part. For example, you may use Problem 1 in your solution of Problem 2, even if you do not attempt Problem 1.

1 Preliminaries

Definition 1 (Independent Set)**.** An *independent set* of a graph $G(V,E)$ is a set $S \subset V$ such that for every $i, j \in S$, $ij \notin E(G)$. Denote $\alpha(G)$ as the size of the largest independent set in G.

(Required) Problem 1. For the following graphs G, determine $\alpha(G)$ and provide an independent set of size $\alpha(G)$. Justify your answer.

(a) $G = C_5$.

(b) $G = Q_3$.

Definition 2 (Graph Vertex Coloring)**.** A *vertex coloring* of a graph $G(V,E)$ is a function $\phi : V(G) \to [n]$ such that whenever $uv \in E(G)$, $\phi(u) \neq \phi(v)$. The *chromatic number* of G, denoted $\chi(G)$, is the smallest $n \in \mathbb{N}$ such that there exists a coloring $\phi : V(G) \to [n]$.

(Required) Problem 2. For the following graphs G, determine $\chi(G)$.

(a) $G = K_n$

(b) $G = C_6$

(c) $G = C_7$

(d) $G = Q_d$

Definition 3 (Clique Cover)**.** Let $G(V,E)$ be a graph. A *clique cover* $\mathcal{C}$ of G is a set of complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_h}$, such that:

- Each K_{n_i} is a subgraph of G;
- No two cliques in $\mathcal{C}$ share any common vertices; and
- Every vertex of G belongs to some clique of $\mathcal{C}$.

The *clique cover number*, denoted $\overline{\chi}(G)$, is the size of the smallest clique cover of $\mathcal{C}$.

Example 1. Consider the cycle graph C_5. Observe that $\mathcal{C} = \{\{1,2\},\{3,4\},\{5\}\}$ forms a clique cover of C_5. In particular, $\{1,2\}$ and $\{3,4\}$ each form a K_2, while $\{5\}$ forms a K_1.

(Advanced) Problem 1. Show that $\overline{\chi}(G) = \chi(\overline{G})$, where $\chi(\overline{G})$ is the chromatic number of $\overline{G}$, the complement of G.

(Required) Problem 3. Show that $\alpha(G) \leq \chi(\overline{G})$.

2 Strong Product

Definition 4 (Strong Product)**.** Let G and H be graphs. Define the *strong product* $G \boxtimes H$ to be the graph with the vertex set $V(G) \times V(H)$. Now two vertices $(i, j), (u, v) \in V(G) \times V(H)$ are adjacent in $G \boxtimes H$ if **both** of the following conditions are satisfied:

(a) $i = u$ or $iu \in E(G)$; **and**

(b) $j = v$ or $jv \in E(H)$.

Example 2. Consider $P_2 \boxtimes P_3$, where P_2 is the path on two vertices. We have the following vertices: $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)$. We provide more detailed explanation for why certain vertices of $P_2 \boxtimes P_2$ are (or are not) adjacent:

- Consider $(1, 1)$ and $(1, 2)$. Observe that the first coordinates are the same (that is, $1 = 1$), so condition (a) of the strong product is satisfied. Now the second coordinates differ. However, $12 \in E(P_2)$. That is, the two vertices in P_2 are adjacent. So the second condition of the strong product is satisfied. Thus, $(1, 1)$ and $(1, 2)$ are adjacent in $P_2 \boxtimes P_2$.
- Consider $(1, 2)$ and $(2, 3)$. While the first coordinates are not equal, we observe that $12 \in E(P_2)$. So condition (a) of the strong product is satisfied. Similarly, we have that $23 \in E(P_3)$. So condition (b) of the strong product is satisfied. Thus, $(1, 2)$ and $(2, 3)$ are adjacent in $P_2 \boxtimes P_3$.
- Consider $(1, 1)$ and $(2, 3)$. While the first coordinates are not equal, we observe that $12 \in E(P_2)$. So condition (a) of the strong product is satisfied. Now consider the second coordinate. Here, we have that $1 \neq 3$; and $13 \notin E(P_3)$. So the condition (b) of the strong product is **not** satisfied.

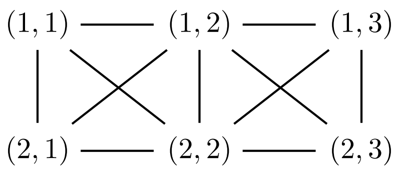

Figure 1. The graph $P_2 \boxtimes P_3$.

(Required) Problem 4. Draw the graph $P_2 \boxtimes C_4$.

Setting. Let Σ be an alphabet. We will be transmitting words over Σ, using a noisy channel (e.g., sending an email, making a phone call, sending a message to satellites in space, the iPhone autocorrect). As our channel is *noisy*, some of the letters you *send* may be *received* as a different letter. We model this situation with a graph $G(V, E)$ (which we call a `confusabiltiy graph`), where $V(G) = \Sigma$. Now two vertices u, v of G are adjacent u can be "confused" with v.

(Required) Problem 5. Let G be a confusability graph over the alphabet Σ. We say that messages $u, v \in \Sigma^n$ are *confusable* if u_i and v_i are confusable (i.e., $u_i v_i \in E(G)$) for all $i \in [n]$. Prove that u and v are confusable if and only if $uv \in E(G^{\boxtimes n})$. Here, $G^{\boxtimes n}$ is the strong product of G with itself n times (e.g., $G^{\boxtimes 3} = G \boxtimes G \boxtimes G$).

(Required) Problem 6. Show that $\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$.

(Required) Problem 7. Show that $\chi(\overline{G \boxtimes H}) \leq \chi(\overline{G})\chi(\overline{H})$.

3 Shannon Capacity

Definition 5 (Shannon Capacity)**.** Let G be a graph. The *Shannon capacity* of G, denoted $\Theta(G)$, is defined as:

$$\Theta(G) := \lim_{n \to \infty} \sqrt[n]{\alpha(G^{\boxtimes n})}.$$

Definition 6 (Supremum)**.** Let $S \subset \mathbb{R}$. The *supremum* of S, denoted $\sup S$, is the *least upper-bound* of S. Formally, suppose that $\sup S = M$. Then for every $\epsilon > 0$, there exists $x \in S$ such that $x > M - \epsilon$.

Example 3. Observe that $\sup [0, 1) = 1$, even though $1 \notin [0, 1)$. Similarly, $\sup [0, 1] = 1$. A set may not have a maximum value. It is helpful to think of the supremum as *what the maximum would be, if the maximum existed.*

For the next problem, you may find the following lemma helpful. You may use the following lemma freely.

Lemma 1 (Feteke's Lemma). Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence that is super-additive; that is, $\{a_n\}_{n\in\mathbb{N}}$ satisfies:

$$a_{n+m} \geq a_n + a_m.$$

Then: $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\sup_{n\in\mathbb{N}} \frac{a_n}{n}$.

(Required) Problem 8. In this problem, we will show that $\Theta(G)$ is *well-defined*; that is, $\Theta(G)$ *exists* for any graph G. In particualr, you are asked to do the following:

(a) Show that $\Theta(G)$ is well-defined, and that: $\Theta(G) = \sup_{n\in\mathbb{N}} \sqrt[n]{(\alpha(G^{\boxtimes n}))}$. [**Hint:** Consider the sequence $a_n := \alpha(G^{\boxtimes n})$. Can you tweak this sequence to a form where Feteke's Lemma applies? You may also want to use Problem 6 .]

(Required) Problem 9. Show that for any graph G, $\alpha(G) \leq \Theta(G)$.

(Advanced) Problem 2. Show that for any graph G, $\Theta(G) \leq \chi(\overline{G})$.

(Required) Problem 10. Show that $\Theta(C_5) \geq \sqrt{5}$.

(Required) Problem 11. Compute $\Theta(G)$ for the following graphs G:

(a) $G = K_n$

(b) $G = \overline{K_n}$

(c) $G = K_{n,n}$

(Open) Problem 1. Determine $\Theta(C_7)$.

(Open) Problem 2. Determine the computational complexity of computing $\Theta(G)$ for an arbitrary graph.