Graph Homomorphisms and Vector Colorings

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Abstract

A graph vertex coloring is an assignment of labels, which are referred to as colors, such that no two adjacent vertices receive the same color. The vertex coloring problem is NP-Complete [8], and so no polynomial time algorithm is believed to exist. The notion of a graph vector coloring was introduced as an efficiently computable relaxation to the graph vertex coloring problem [7]. In [6], the authors examined the highly symmetric class of 1-walk regular graphs, characterizing when such graphs admit unique vector colorings. We present this characterization, as well as several important consequences discussed in [5, 6]. By appealing to this characterization, several important families of graphs, including Kneser graphs, Quantum Kneser graphs, and Hamming graphs, are shown to be uniquely vector colorable. Next, a relationship between locally injective vector colorings and cores is examined, providing a sufficient condition for a graph to be a core. As an immediate corollary, Kneser graphs, Quantum Kneser graphs, and Hamming graphs are shown to be cores. We conclude by presenting a characterization for the existence of a graph homomorphism between Kneser graphs having the same vector chromatic number. The necessary condition easily generalizes to Quantum Kneser graphs, simply by replacing combinatorial expressions with their quantum analogues.

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Chapter 1

Introduction

Graph vertex coloring, often referred to as simply *graph coloring*, is a special case of graph labeling. Each vertex of a graph G is assigned a label or *color*, such that no two adjacent vertices receive the same color. Formally, an m-coloring of a graph G is a graph homomorphism $\varphi : V(G) \to V(K_m)$. It is of particular interest to optimize the parameter m; that is, to find the smallest $m \in \mathbb{Z}^+$ such that there exists a graph homomorphism $\varphi : V(G) \to V(K_m)$. Here, the smallest such m is referred to as the *chromatic number* of G, dentoed $\chi(G)$. For parameters $m \geq 3$, deciding if a graph has an m-coloring is one of Richard Karp's 21 NP-Complete problems from 1972 [8].

In [7], the notion of a vector coloring was introduced as a relaxation of the graph coloring problem. For real-valued parameters of $t \geq 2$, a vector t-coloring is an assignment of unit vectors in $\mathbb{R}^d$ to the vertices of the graph, such that the vectors v_i and v_j assigned to adjacent vertices i and j respectively, satisfy:

$$\langle v_i, v_j \rangle \leq -\frac{1}{t-1}. \tag{1.1}$$

The *vector chromatic number* of a graph G, denoted $\chi_v(G)$, is the smallest such t that G admits a vector t-coloring. A vector t-coloring is said to be strict if condition (1.1) holds with equality for all pairs of adjacent vertices i and j. The *strict vector chromatic number* of a graph G, denoted $\chi_{sv}(G)$, is the smallest such t that G admits a strict vector t-coloring. Both $\chi_v(G)$ and $\chi_{sv}(G)$ can be approximated arbitrarily close to their actual values in polynomial time. Thus, the notion of graph vector coloring finds immediate motivation in computational complexity, serving as an efficiently computable lower bound for the chromatic number of a graph [5, 6].

Graph vector coloring also has applications to information theory. Suppose Σ is a finite set of letters, which is referred to as an *alphabet*. A message composed of words over Σ is sent over a noisy channel, in which some of the letters may be confused. The *confusability graph* $G(V, E)$ of the given channel has vertex set Σ, where two distinct elements $i, j \in \Sigma$ are adjacent if i and j can be confused. Observe that the independent sets of G are precisely the sets of pairwise non-confusable characters. So the independence number $\alpha(G)$ is the maximum number of non-confusable characters that can be sent over the given channel. The *Shannon capacity* of a graph G, denoted $\Theta(G)$, is defined as: $\Theta(G) := \sup\limits_{k \geq 1} \sqrt[k]{\alpha(G^k)}$, where G^k is the *strong product* of G with itself k times. The complexity of computing the Shannon capacity of a graph remains an open problem. It is well known, for example, that $\Theta(C_5) = \sqrt{5}$. However, even determining $\Theta(C_7)$ remains an open problem [10]. In 1979, Lovász introduced a graph parameter, known as the *Lovász theta number*, $\vartheta(G)$, with the explicit goal of estimating the Shannon capaicty [10]. It is well known that $\chi_{sv}(G) = \vartheta(\overline{G})$, where $\overline{G}$ denotes the complement of G [7]. So the strict vector chromatic number serves as an efficiently computable upper bound for the Shannon capacity.

This monograph surveys the results of [5, 6]. Chapter 2 serves to introduce preliminary definitions and lemmata. Next, Chapter 3 begins by presenting the result of [6] characterizing 1-walk regular graphs that have unique vector colorings. To this end, we examine graph embeddings constructed from the eigenvectors corresponding to the smallest eigenvalue of the adjacency matrix. Such frameworks are referred to as *least eigenvalue frameworks*. In particular, the least eigenvalue framework of 1-walk regular graph always provides an optimal vector coloring. Furthermore, the (strict) vector chromatic number of such graphs depends only on the degree of the graph and smallest eigenvalue of the adjacency matrix. Chapter 3 concludes by demonstrating that Kneser graphs, Quantum Kneser graphs, and Hamming graphs are uniquely vector colorable [5].

Chapter 4 presents the relationship between vector colorings and cores, which is established in [5]. Intuitively, a core is a graph G in which every homomorphism $\varphi : V(G) \to V(G)$ is an automorphism. Using this relationship, several families of graphs, including Kneser graphs, Quantum Kneser graphs, and Hamming graphs, are easily shown to be cores. Chapter 4 concludes by characterizing the existence of a graph homomorphism between Kneser graphs having the same vector chromatic number. The necessary condition easily generalizes to Quantum Kneser graphs, simply by replacing combinatorial expressions with their quantum analogues.

Chapter 2

Definitions and Preliminaries

Key definitions from graph theory, linear algebra, and algebraic combinatorics will first be introduced. Much of this material can be found in standard references such as [2, 3, 9, 15]. After introducing definitions, preliminary lemmata will be presented.

Notation 2.1. Let $n \in \mathbb{N}$. Denote $[n] := \{1, 2, \ldots, n\}$, with the convention that $[0] = \emptyset$.

Notation 2.2. Let S be a set, and let $k \in \mathbb{N}$. Denote $\binom{S}{k}$ as the set of k-element subsets of S.

Notation 2.3. Let p be prime, and let $q = p^\alpha$ for some $\alpha \in \mathbb{Z}^+$. Denote $\mathbb{F}_q$ as the finite field of order q.

2.1 Linear Algebra and Algebraic Combinatorics

Definition 2.4. Let V be a finite dimensional vector space, with dimension n. For each $i \in [n]$, denote the ith standard basis vector $e_i \in V$ to be the vector whose ith coordinate is 1 and all other components are 0. The set $\{e_1, e_2 \ldots, e_n\}$ is referred to as the *standard basis*.

Definition 2.5. Direct Sum of Matrices Let M_1, M_2 be matrices. The *direct sum* $M_1 \oplus M_2$ is the matrix:

$$M_1 \oplus M_2 := \begin{pmatrix} M_1 & O \\ O & M_2 \end{pmatrix}.$$

Definition 2.6 (Line)**.** Let V be a vector space. A *line* in V is a one-dimensional subspace of V.

Definition 2.7 (Skew Lines)**.** Two lines are said to be *skew* if their intersection is the trivial subspace, $\{0\}$.

Notation 2.8. Let $u, v \in \mathbb{R}^n$. The standard inner product on $\mathbb{R}^n$ is given by:

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i.$$

Definition 2.9 (Unit Vector)**.** A vector $v \in \mathbb{R}^n$ is said to be a *unit vector* if:

$$\sum_{i=1}^{n} |v_i|^2 = 1.$$

Definition 2.10 (Orthogonal Transformation)**.** A linear transformation $T : V \to V$ on an inner product space V over $\mathbb{R}$ is said to be *orthogonal* if for every $u, v \in V$:

$$\langle u, v \rangle = \langle T(u), T(v) \rangle.$$

Definition 2.11 (Orthogonal Complement)**.** Let V be an inner product space, and let W be a subspace of V. The *orthogonal complement* of W is the set:

$$W^{\perp} = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in W\}.$$

Definition 2.12 (Convex Hull)**.** Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. The *convex hull* of $v_1, v_2, \ldots, v_k$ is the set:

$$\operatorname{conv}(\{v_1, \ldots, v_k\}) = \left\{ \sum_{i=1}^{k} \lambda_i v_i : \sum_{i=1}^{k} \lambda_i = 1, \text{ and } \lambda_i \geq 0 \text{ for all } i \in [k] \right\}.$$

Definition 2.13 (Affine Independence)**.** The vectors $v_0, v_1, \ldots, v_k \in \mathbb{R}^n$ are *affinely independent* if $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly independent.

Definition 2.14 (Simplex)**.** Let $n \in \mathbb{N}$. Let $v_0, v_1, \ldots, v_{n+1} \in \mathbb{R}^{n+1}$ be affinely independent unit vectors, such that the angle subtended by any two distinct v_i and

v_j through the origin is $\arccos(-1/n)$. The *simplex* $\Delta^n \subset \mathbb{R}^{n+1}$ centered at the origin is given by:

$$\Delta^n = \operatorname{conv}(\{v_0, v_1, \ldots, v_{n+1}\}).$$

Note that for any two distinct vertices u and v of the simplex Δ^n, $\langle u, v\rangle = -\frac{1}{n}$. This property will be leveraged later to show that the vector chromatic number of a graph is a lower bound for the chromatic number.

Definition 2.15 (Gram matrix)**.** The *Gram matrix* of a set of vectors $v_1, \ldots, v_n$, denoted $\operatorname{Gram}(v_1, \ldots, v_n)$, is the $n \times n$ matrix with ij-entry equal to $\langle v_i, v_j\rangle$.

The Gram matrix is positive semidefinite, and has rank equal to the dimension of $\operatorname{span}(v_1, \ldots, v_n)$. [6]

Definition 2.16 (Hermitian Matrix)**.** An $n \times n$ matrix M is *Hermitian* if M is equal to its conjugate transpose. That is, $M_{ij} = \overline{M_{ji}}$ for all $i, j \in [n]$, where $\overline{M_{ji}}$ denotes the complex conjugate.

Definition 2.17 (Positive Semidefinite Matrix)**.** An $n \times n$ Hermitian matrix M is *positive semidefinite* if all the eigenvalues of M are non-negative.

Notation 2.18. Let $n \in \mathbb{N}$. Denote $\mathcal{S}^n$ as the set of $n \times n$ symmetric matrices over $\mathbb{R}$. Similarly, denote $\mathcal{S}^n_+$ as the elements of $\mathcal{S}^n$ that are positive semidefinite.

Definition 2.19 (Schur Product)**.** The *Schur product* of two matrices $X, Y \in \mathcal{S}^n$, denoted $X \circ Y$, is given by: $(X \circ Y)_{ij} = X_{ij}Y_{ij}$ for all $i, j \in [n]$.

Notation 2.20. Let $n \in \mathbb{N}$. The Symmetry group of degree n is denoted $\operatorname{Sym}(n)$.

Definition 2.21 (Grassmanian)**.** Let $\mathbb{F}$ be a field, and let $n, k \in \mathbb{N}$. The *Grassmanian* $\operatorname{Gr}_n(k, \mathbb{F})$ is the set of all k-dimensional subspaces of the vector space $\mathbb{F}^n$.

Tools from quantum combinatorics will next be introduced, which provide a generalization of combinatorics on set systems to the linear algebraic setting.

Definition 2.22 (Quantum Integer)**.** Let $n \in \mathbb{N}$. The *quantum integer* $[n]_x$ is the function:

$$[n]_x := \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}.$$

Definition 2.23 (Quantum Factorial)**.** Let $n \in \mathbb{N}$. The *quantum factorial* $[n!]_x$ is the function:

$$[n!]_x = \prod_{i=1}^{n} [i]_x = \prod_{i=1}^{n} \frac{x^i - 1}{x - 1}.$$

Definition 2.24 (Quantum Binomial Coefficient)**.** Let $n, k \in \mathbb{N}$. The *quantum binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_x$ is the function:

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \frac{[n!]_x}{[k!]_x [(n-k)!]_x}.$$

Let p be prime, and let $q = p^a$ for some $a \in \mathbb{N}$. It is well-known that $\begin{bmatrix} n \\ k \end{bmatrix}_q = |\text{Gr}_n(k, \mathbb{F}_q)|$. Note as well that $\begin{bmatrix} n \\ 1 \end{bmatrix}_x = [n]_x$. Thus, $[n]_q$ counts the number of lines, or one-dimensional subspaces, of $\mathbb{F}_q^n$. It is also worth noting that as $q \to 1$, $[n]_q \to n$, $[n!]_q \to n!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q \to \binom{n}{k}$. [9] This close relationship will be used to generalize results about set systems to a linear algebraic setting. Not surprisingly, the proofs about set systems generalize by simply replacing the classical combinatorial terms with their quantum analogues.

Group actions will next be introduced. The notion of a group action is a particularly powerful and useful notions from algebra, which formalizes the notion of symmetry. Intuitively, a group action is a discrete dynamical process on a set of elements that partitions the set. The structure and number of these equivalence classes provide important insights in algebra, combinatorics, and graph theory.

Definition 2.25 (Group Action)**.** Let Γ be a group, and let S be a set. A *group action* is a function $\cdot : \Gamma \times S \to S$ satisfying the following:

(a) $1 \cdot s = s$ for all $s \in S$; and

(b) $g \cdot (h \cdot s) = (gh) \cdot s$ for all $g, h \in \Gamma$ and all $s \in S$.

Here, gh is understood to be the group operation of Γ. Note that a group action $\cdot : \Gamma \times S \to S$ induces a group homomorphism $\varphi : \Gamma \to \mathrm{Aut}(S)$.

Example 2.26. Let $n \in \mathbb{N}$. The group $\mathrm{Sym}(n)$ acts on $[n]$ in the following manner: for $\sigma \in \mathrm{Sym}(n)$ and $i \in [n]$, $\sigma \cdot i \mapsto \sigma(i)$.

Example 2.27. Let Γ be a group. The *natural left action* of Γ on itself is the map $\cdot : \Gamma \times \Gamma \to \Gamma$, where $g \cdot h \mapsto gh$. Here, gh is understood to be the product of g and h according to the operation of Γ.

Definition 2.28 (Orbit)**.** Let Γ be a group, acting on the set S. The *orbit* of an element $s \in S$ is the set $\mathcal{O}(s) = \{g \cdot s : g \in \Gamma\}$.

The orbits of a group action partition the set S upon which the group Γ acts, and so the orbit relation forms an equivalence relation.

The next term to be introduced is a transitive action. A group action is transitive if for every pair of elements i and j in the set S, there exists an element g of the group such that $g \cdot i \mapsto j$.

Definition 2.29 (Transitive Action)**.** Let Γ be a group, and let S be a set. The group action $\cdot : \Gamma \times S \to S$ is said to be *transitive* if there exists a single orbit, which is the entire set S, under this action.

Example 2.30. The group action in Example 2.26 is indeed transitive. Again, let $n \in \mathbb{N}$. Let $i, j \in [n]$. If $i = j$, the identity function will map $i \mapsto j$. If $i \neq j$, the permutation $(i, j) \in \mathrm{Sym}(n)$ will map $i \mapsto j$.

Definition 2.31 (Character)**.** Let Γ be a group, and let $\mathbb{C}$ be a field. A *character* is a group homomorphism $\varphi : \Gamma \to \mathbb{C}^{\times}$.

Definition 2.32 (Simple Graph)**.** A *simple graph* $G(V,E)$ consists of a set of vertices V, along with a set of edges $E \subset \binom{V}{2}$. An edge $\{u,v\}$ will be denoted as uv. The adjacency relation $\sim$ is a binary relation on V, such that $u \sim v$ if and only if $uv \in E(G)$. The relation $u \sim v$ is read as "u is adjacent to v." If multiple graphs are being considered, the graph in question may be subscripted on the relation. That is, $u \sim_G v$ refers to the adjacency relation with respect to the graph G. The relation $\simeq$ is a binary relation on V, where $u \simeq v$ denotes that $u \sim v$ or $u = v$.

Unless otherwise stated, all graphs are assumed to be simple and will be referred to as *graphs.* In order to avoid ambiguity, the vertex set of the graph G will frequently be denoted as $V(G)$. Similarly, the edge set of the graph G will be denoted as $E(G)$.

Definition 2.33. Let $G(V,E)$ be a graph, and let $v \in V(G)$. The *neighborhood* of v is the set: $N(v) = \{u : uv \in E(G)\}$. The *degree* of v, denoted $\deg(v)$, is $|N(v)|$.

Several important classes of graphs will next be introduced.

Definition 2.34 (Regular Graph)**.** A graph $G(V,E)$ is said to be *regular* if every vertex has the same degree d. Here, d is referred to as the *degree* of G.

Definition 2.35 (Complete Graph)**.** Let $n \in \mathbb{N}$. The *complete graph* on n vertices, denoted K_n, has the vertex set $V(K_n) = [n]$ with the edge set $E(K_n) = \binom{[n]}{2}$.

Definition 2.36 (Cycle Graph)**.** Let $n \in \mathbb{Z}^+$ with $n \geq 3$. The *cycle graph* on n verties, denoted C_n, has the vertex set $V(C_n) = [n]$ with the edge set:

$$E(C_n) = \{\{i, i+1\} : i \in [n-1]\} \cup \{\{1,n\}\}.$$

Definition 2.37 (Hypercube)**.** Let $d \in \mathbb{N}$. The *hypercube of degree d*, denoted Q_d, has vertex set $\mathbb{F}_2^d$. Two vertices $(v_1, v_2, \ldots, v_d)$ and $(w_1, w_2, \ldots, w_d)$ in Q_d are adjacent if and only if they differ in precisely one position.

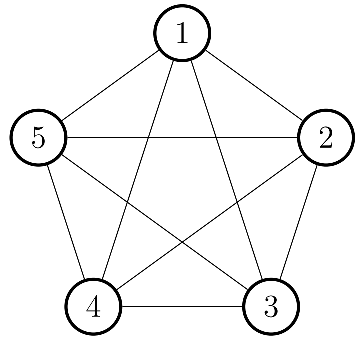

Figure 2.1: The complete graph on 5 vertices, K_5.

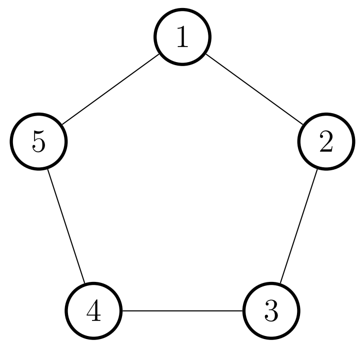

Figure 2.2: The cycle graph on 5 vertices, C_5.

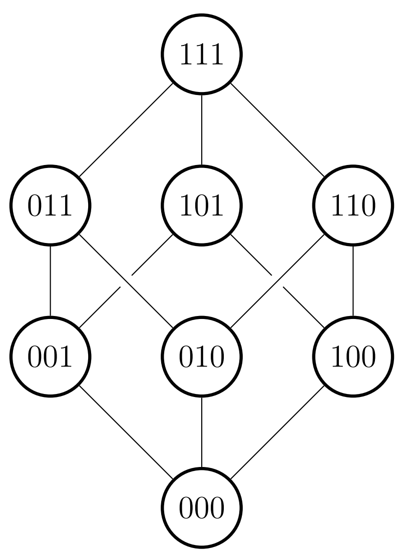

Figure 2.3: The hypercube of degree 3, Q_3.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 2.4: The Adjacency Matrix of K_5

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Example 2.38. The graphs K_n, C_n, and Q_d are all regular. Observe that $\deg(K_n) = n-1$, $\deg(C_n) = 2$, and $\deg(Q_d) = d$.

Definition 2.39 (Bipartite Graph)**.** A graph $G(V, E)$ is said to be *bipartite* if there exist nonempty, disjoint sets $A, B \subset V(G)$ such that $A \cup B = V(G)$ and $E(G) \subset \{ab : a \in A, b \in B\}$.

Example 2.40. The graph C_{2n} is bipartite for all $n \geq 2$, and Q_d is bipartite for all $d \in \mathbb{N}$.

Definition 2.41 (Adjacency Matrix)**.** Let $G(V, E)$ be a graph. The *adjacency matrix* of G, dentoed $A(G)$ or A when G is understood, is the $|V| \times |V|$ matrix where for all vertices $i, j \in V(G)$:

$$A_{ij} = \begin{cases} 1: & ij \in E(G) \\ 0: & \text{Otherwise.} \end{cases}$$

Definition 2.42 (Graph Homomorphism)**.** Let G and H be graphs. A *graph homomorphism* is a function $\varphi : V(G) \to V(H)$ such that if $ij \in E(G)$, then $\varphi(i)\varphi(j) \in E(H)$. An *endomorphism* is a graph homomorphism $\varphi : V(G) \to V(G)$. Denote $\text{Hom}(G, H)$ as the set of graph homomorphisms from G to H. When $G = H$, the set $\text{Hom}(G, G)$ is denoted $\text{End}(G)$.

Definition 2.43 (Graph Isomorphism)**.** Let G and H be graphs. A *graph isomorphism* is a bijection $\varphi : V(G) \to V(H)$ that is also a graph homomorphism. The graphs G and H are *isomorphic*, dentoed $G \cong H$, if there exists an isomorphism $\varphi : V(G) \to V(H)$. When $G = H$, φ is referred to as a *graph automorphism.* The automorphisms of a graph G form a group, which is denoted $\mathrm{Aut}(G)$.

Common examples of graph homomorphisms include graph colorings.

Definition 2.44 (Graph Coloring)**.** Let G be a graph, and let $m \in \mathbb{N}$. An m-coloring of a graph G is a graph homomorphism $\varphi : V(G) \to V(K_m)$. It is of particular interest to optimize the parameter m; that is, to find the smallest $m \in \mathbb{N}$ such that there exists a graph homomorphism $\varphi : V(G) \to V(K_m)$. Here, the smallest such m as the *chromatic number* of G, dentoed $\chi(G)$.

For parameters $m \geq 3$, deciding if a graph has an m-coloring is one of Richard Karp's 21 NP-Complete problems from 1972. [8]

Example 2.45. It is well known that a graph G is bipartite if and only if $\chi(G) = 2$. [15]

Example 2.46. Consider the complete graph K_n. As K_n has n vertices, any assignment of $n-1$ or fewer colors to $V(K_n)$ will result in two distinct vertices receiving the same color. As every pair of vertices in K_n are adjacent, it follows that $\chi(K_n) \geq n$. Now the identity map on $V(K_n)$ is certainly a graph coloring. So $\chi(K_n) = n$.

Example 2.47. Let $n \geq 3$ be an integer, and consider the graph C_n. If n is even, C_n is bipartite, in which case $\chi(C_n) = 2$. If n is odd, $\chi(C_n) = 3$.

Definition 2.48 (Core)**.** A graph $G(V, E)$ is said to be a *core* if $\mathrm{End}(G) = \mathrm{Aut}(G)$.

Example 2.49. Common examples of cores include odd cycles. [3]

The next class of graph to be introduced is the Cayley graph. Intuitively, a Cayley graph provides a combinatorial means of visualizing a group Γ's operation. Formally, the Cayley graph is defined as follows.

Definition 2.50 (Cayley Graph)**.** Let Γ be a group, and let $S \subset \Gamma$ such that $S = S^{-1}$ and $1 \notin \Gamma$. The *Cayley graph* $\text{Cay}(\Gamma, S)$ has vertex set Γ. Two elements $g, h \in \Gamma$ are adjacent in $\text{Cay}(\Gamma, S)$ if and only if there exists $s \in S$ such that $gs = h$.

Example 2.51. Let $n \in \mathbb{Z}^+$ with $n \geq 3$, and consider the group $\mathbb{Z}_n$. Let $S = \{\pm 1\} \subset \mathbb{Z}_n$. So $C_n \cong \text{Cay}(\mathbb{Z}_n, S)$.

Example 2.52. Let $n \in \mathbb{Z}^+$, and consider the group $\mathbb{Z}_n$. Let $S = \mathbb{Z}_n \setminus \{0\}$. The graph $K_n \cong \text{Cay}(\mathbb{Z}_n, S)$.

Example 2.53. Let $n \in \mathbb{N}$, and consider the group $\mathbb{F}_2^n$. Let $S = \{e_1, e_2, \ldots, e_n\}$. The graph $Q_n \cong \text{Cay}(\mathbb{F}_2^n, S)$.

Definition 2.54 (Vertex Transitive Graph)**.** A graph $G(V, E)$ is said to be *vertex transitive* if $\text{Aut}(G)$ acts transitively on $V(G)$; that is, if for every $u, v \in \text{Aut}(G)$, there exists $\sigma \in V(G)$ such that $\sigma(u) = v$.

Example 2.55. It is well-known that Cayley graphs are vertex transitive. [3]

Definition 2.56 (1-Walk Regular Graph)**.** Let G be a graph with adjacency matrix A. G is said to be *1-walk regular* if for every $\ell \in \mathbb{N}$, there exist constants a_ℓ, b_ℓ such that: $A^\ell \circ I = a_\ell I$, and $A^\ell \circ A = b_\ell A$.

Equivocally, a 1-walk regular graph $G(V, E)$ satisfies the following:

- For every vertex v and every $i \in \mathbb{N}$, the number of closed walks of length i starting at v depends only on i and not v; and
- For every edge uv and every $k \in \mathbb{N}$, the number of walks of length k between u and v depends only on k.

So 1-walk regular graphs exhibit a high degree of symmetry. Necessarily, 1-walk regular graphs are regular. An important class of graph are 1-walk regular, including vertex-transitive graphs. [6]

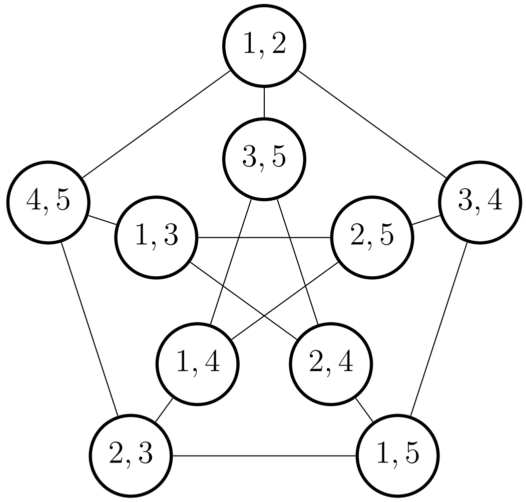

Figure 2.5: The Kneser Graph KG$(5, 2)$.

Definition 2.57 (Kneser Graph)**.** Let $n, k \in \mathbb{N}$, with $n \geq k$. The *Kneser graph* KG(n, k) has vertex set $\binom{[n]}{k}$, and two vertices S and T are adjacent if and only if S and T are disjoint.

Example 2.58. Perhaps the most famous example of a Kneser graph is the Petersen graph, which is defined as KG$(5, 2)$.

Definition 2.59 (Quantum Kneser Graph)**.** Let $q = p^\alpha$, for some prime p and $\alpha \in \mathbb{Z}^+$. For $n, k \in \mathbb{N}$ with $n \geq k$, define the *Quantum Kneser graph*, or *q-Kneser Graph*, q-KG(n, k) as the graph whose vertices are the k-dimensional subspaces of $\mathbb{F}_q^n$. Two vertices S and T in q-KG(n, k) are adjacent precisely when $S \cap T = \{0\}$.

Intuitively, the q-Kneser graph generalizes the Kneser graph to the linear algebraic setting. The Kneser graph relates k-element subsets of $[n]$ that intersect trivially. The building blocks of each subset are the elements of $[n]$. The q-Kneser graph relates k-dimensional subspaces which intersect trivially. As every subspace contains the additive identity 0, two subspaces S and T intersect trivially if $S \cap T = \{0\}$. Now if S is a k-dimensional subspace of $\mathbb{F}_q^n$ and $v \in S$ is a vector, then span$(v) \subset S$.

So 1-dimensional subspaces, or lines, are the building blocks of the k-dimensional subspaces of $\mathbb{F}_q^n$.

2.3 Vector Colorings

Definition 2.60. Let $t \geq 2$ be a real number, and let $d \in \mathbb{N}$. Let $\mathcal{S}_t^d$ be the graph whose vertices are the unit vectors in $\mathbb{R}^d$. Two unit vectors u and v in $\mathbb{R}^d$ are adjacent in $\mathcal{S}_t^d$ precisely when:

$$\langle u, v \rangle \leq -\frac{1}{t-1}.$$

Definition 2.61 (Vector Coloring)**.** Let $t \geq 2$ be a real number. Let G be a non-empty graph with n vertices, and denote S as the set of unit vectors in $\mathbb{R}^d$. A *vector t-coloring* of a graph G is a graph homomorphism $\varphi : V(G) \to V(\mathcal{S}_t^d)$. The *vector chromatic number* of G, denoted $\chi_v(G)$, is the smallest real number $t \geq 2$ such that a vector t-coloring of G exists. The *value* of a vector coloring φ is the smallest $t \geq 2$ such that there exists a graph homomorphism from G to $\mathcal{S}_t^d$. A vector coloring is *optimal* if its value is $\chi_v(G)$. The vector coloring φ is *strict* if for every $i \sim j$ in $V(G)$, that:

$$\langle \varphi(i), \varphi(j) \rangle = -\frac{1}{t-1}.$$

The *strict vector chromatic number* of G is denoted $\chi_{sv}(G)$.

Every strict vector coloring is clearly a vector coloring. It follows that $\chi_v(G) \leq \chi_{sv}(G)$. It will first be established that for any graph G, $\chi_{sv}(G) \leq \chi(G)$. This provides the relation that $\chi_v(G) \leq \chi_{sv}(G) \leq \chi(G)$.

Lemma 2.62. *For any non-empty graph G, $\chi_{sv}(G) \leq \chi(G)$.*

Proof. Suppose that $\chi(G) = m$, and fix a m-coloring $\varphi : V(G) \to V(K_m)$. Consider now the m-dimensional simplex Δ^m centered at the origin, with vertices labeled $t_1, \dots, t_m$. A strict vector m-vector coloring $\tau : V(G) \to \{t_1, \dots, t_m\}$ will be constructed in the following manner. If $\varphi(v) = i$, set $\tau(v) = t_i$. Now as Δ^m is centered at the origin, $\langle t_i, t_j \rangle = -\frac{1}{m}$ whenever $i \neq j$. By the construction of τ, if $u \sim v$, then $\tau(u) \neq \tau(v)$. Thus, for all $u \sim v$, $\langle \tau(u), \tau(v) \rangle = -\frac{1}{m}$. So τ is a strict m-vector coloring of G. $\square$

Using semidefinite programming both the vector chromatic number $\chi_v(G)$ and the strict vector chromatic number $\chi_{sv}(G)$ are polynomial time computable, up to an arbitrarily small error in the inner products. This is formalized as follows. Let $\epsilon > 0$. If a (strict) vector t-coloring exists, then a $(t + \epsilon)$ (strict) vector coloring can be constructed in polynomial time, with respect to the number of vertices n and $\log(1/\epsilon)$. [7]

The semidefinite program for $\chi_v(G)$ is given below:

$$\chi_v(G) = \min_{t \geq 2} t \qquad \text{subject to:} \tag{2.1}$$

$$\langle v_i, v_j \rangle \leq -\frac{1}{t-1} \text{ for all } ij \in E(G) \tag{2.2}$$

$$\langle v_i, v_i \rangle = 1 \text{ for all } i \in V(G). \tag{2.3}$$

Similarly, the semidefinite program for $\chi_{sv}(G)$ arises by replacing the inequality in constraint (2.2) with equality. This yields the following semidefinite program:

$$\chi_v(G) = \min_{t \geq 2} t \qquad \text{subject to:} \tag{2.4}$$

$$\langle v_i, v_j \rangle = -\frac{1}{t-1} \text{ for all } ij \in E(G) \tag{2.5}$$

$$\langle v_i, v_i \rangle = 1 \text{ for all } i \in V(G). \tag{2.6}$$

The following examples provide $\chi_v(G)$ and $\chi_{sv}(G)$ for two common classes of graphs.

Example 2.63. Suppose G is a non-empty bipartite graph. Recall that $2 \leq \chi_v(G) \leq \chi_{sv}(G) \leq \chi(G)$. Now as G is bipartite, $\chi(G) = 2$. Thus, $\chi_v(G) = \chi_{sv}(G) = 2$.

Example 2.64. Suppose $G = K_n$ for some $n \geq 2$. Recall that $\chi(G) = n$. So $\chi_v(G) \leq \chi_{sv}(G) \leq n$. Suppose to the contrary that $\chi_v(G) \neq n$. Let φ be a vector

t-coloring of G for $t < n$. So there exist two vertices v_i, v_j such that $\varphi(v_i) = \varphi(v_j)$. As G is complete, $v_i \sim v_j$. So $\langle \varphi(v_i), \varphi(v_j) \rangle \geq 0$, contradicting the assumption that φ is a vector coloring of G. Thus, $\chi_v(G) = \chi_{sv}(G) = n$.

The study of graph homomorphisms will be related to vector colorings in the following manner. Suppose there exists a vector t-coloring φ_1 of H, and a graph homomorphism $\varphi_2 : V(G) \to V(H)$. Then $\varphi_1 \circ \varphi_2$ is a vector t-coloring of G. In order to establish a homomorphism from $G \to H$, graphs whose vector colorings have particular structure will be considered. This section will be concluded with two helpful lemmas.

Lemma 2.65. *Let G and H be graphs, with $\chi_v(G) = \chi_v(H)$. If φ_1 is an optimal vector coloring of H and $\varphi_2 : V(G) \to V(H)$ is a graph homomorphism, then $\varphi_1 \circ \varphi_2$ is an optimal vector coloring of G.*

Proof. Let $t := \chi_v(G) = \chi_v(H)$. As φ_1 is an optimal t-coloring of H, it follows that:

$$\langle \varphi_1(i), \varphi_1(j) \rangle \leq -\frac{1}{t-1} \text{ for all } i \sim_H j.$$

Now let $u, v \in V(G)$ such that $u \sim_G v$. As φ_2 is a graph homomorphism, $\varphi_2(u) \sim_H \varphi_2(v)$. Thus:

$$\langle (\varphi_1 \circ \varphi_2)(u), (\varphi_1 \circ \varphi_2)(v) \rangle \leq -\frac{1}{t-1}.$$

So $\varphi_1 \circ \varphi_2$ is a vector t-coloring of G. As $\chi_v(G) = t$, $\varphi_1 \circ \varphi_2$ is an optimal t-coloring of G. $\square$

Lemma 2.66. *Let G and H be graphs such that $\chi_v(G) = \chi_v(H) = t$, and suppose that every optimal vector coloring of G is injective. Then the following conditions hold:*

(a) Any homomorphism $\varphi : V(G) \to V(H)$ is injective.

(b) If additionally, every optimal vector coloring ψ of G satisfies:

$$\langle \psi(u), \psi(v) \rangle \leq \frac{-1}{t-1} \text{ if and only if } u \sim_G v,$$

then any homomorphism $\varphi : V(G) \to V(H)$ is an isomorphism to an induced subgraph of H.

Proof. (a) Suppose to the contrary that there exists a non-injective homomorphism $\varphi_1 : V(G) \to V(H)$. Let $u, v \in V(G)$ be distinct such that $\varphi_1(u) = \varphi_1(v)$. By Lemma 2.65, the composition of φ_1 with any optimal vector coloring of H yields an optimal vector coloring of G. However, u and v are assigned the same vector, contradicting the assumption that every optimal vector coloring of G is injective.

(b) Let $\varphi : V(G) \to V(H)$ be a homomorphism. Suppose to the contrary that φ is not an isomorphism to an induced subgraph of H. By (a), φ is injective. As φ is not an isomorphism, there exist vertices $u, v \in V(G)$ such that $u \not\sim_G v$ but $\varphi(u) \sim_H \varphi(v)$. Let τ be an optimal vector coloring of H. By Lemma 2.65, $\varphi \circ \tau$ is an optimal vector coloring of G. Thus:

$$\langle (\varphi \circ \tau)(u), (\varphi \circ \tau)(v) \rangle \leq \frac{-1}{t-1}.$$

However, $u \not\sim_G v$, a contradiction.

□

Chapter 3

Vector Colorings of Graphs

3.1 Unique Vector Colorings of 1-Walk Regular Graphs

A graph G with chromatic number m is said to be uniquely m-colorable if it has a m-coloring $\varphi : V(G) \to V(K_m)$; and for any other m-coloring $\tau : V(G) \to V(K_m)$, there exists a permutation $\sigma \in \text{Sym}(m)$ such that $\varphi = \sigma \circ \tau$. In a similar manner, applying an orthogonal transformation to a (strict) vector coloring yields another (strict) vector coloring. Formally, let $\varphi : V(G) \to V(\mathcal{S}_t^d)$ be a t-vector coloring, and let $U : \mathbb{R}^d \to \mathbb{R}^\ell$ be an orthogonal transformation. As the map U preserves the inner product, it follows that:

$$\langle U(\varphi(i)), U(\varphi(j)) \rangle = \langle \varphi(i), \varphi(j) \rangle \leq \frac{-1}{t-1} \text{ for all } i \sim j.$$

So the map $U \circ \varphi$ is also a vector coloring of the same value. This is the analogue of permuting colors in a standard graph coloring. The notion of unique vector colorability is captured using using the Gram matrix.

Definition 3.1. The graph G is said to be *uniquely (strict) vector colorable* if for any two optimal strict vector colorings $\varphi : V(G) \to V(\mathcal{S}_t^d)$ and $\tau : V(G) \to V(\mathcal{S}_t^\ell)$, we have that:

$$\text{Gram}(\varphi(v_1), \ldots, \varphi(v_n)) = \text{Gram}(\tau(v_1), \ldots, \tau(v_n)).$$

Here, φ is said to be the *unique optimal vector coloring* of G, and φ and τ are said to be *congruent*.

We now explore necessary and sufficient conditions for 1-walk regular graphs to be uniquely vector colorable. The first step is to introdue the *canonical* vector coloring. Let G be a 1-walk regular graph, and let A be the adjacency matrix of G. Let P be the $n \times d$ matrix, whose columns form an orthonormal basis for the eigenspace associated with the smallest eigenvalue $\lambda_{\min}$ of A. Let p_i be the ith row of P. The map $\varphi : V(G) \to \mathbb{R}^d$ sending $\varphi(i) = \sqrt{\frac{n}{d}} p_i$ is referred to as the *canonical* vector coloring. The canonical vector coloring serves as the basis for comparing other vector colorings of G. It will later be shown that the canonical vector coloring is in fact an optimal strict vector coloring of G. It will first be established that the map φ is indeed a strict vector coloring of G.

Lemma 3.2. *Let G be a 1-walk regular graph with n vertices, and let A be the adjacency matrix of G. Let P be the $n \times d$ matrix, whose columns form an orthonormal basis associated with the smallest eigenvalue $\lambda_{\min}$ of A. Let p_i be the ith row of P. The map $\varphi : V(G) \to \mathbb{R}^d$ sending $\varphi(i) = \sqrt{\frac{n}{d}} p_i$ is a strict vector coloring of G.*

Proof. It will be shown that $\langle p_i, p_i \rangle = d/n$ for all $i \in [n]$, and that there exists $b < 0$ such that for all $i \sim j$, $\langle p_i, p_j \rangle = b$. Let $E_{\lambda_{\min}} := PP^T$ be the orthogonal projector onto the $\lambda_{\min}$ eigenspace of G. Denote:

$$Z := \prod_{\lambda \neq \lambda_{\min}} \frac{1}{\lambda_{\min} - \lambda}(A - \lambda I). \tag{3.1}$$

It will first shown that $Z = E_{\lambda_{\min}}$. Let $\beta = \{\beta_1, \ldots, \beta_n\}$ be an orthonormal basis composed of eigenvectors of A. Suppose $v \in \beta$ is not in the eigenspace of $\lambda_{\min}$. So v is in the eigenspace associated with some eigenvalue $\tau \neq \lambda_{\min}$. Thus, $(A - \tau I)v = 0$. It follows that $Zv = 0$. Now suppose instead v is in the eigenspace associated with $\lambda_{\min}$. It follows that:

$$
\begin{aligned}
Zv &= \prod_{\lambda \neq \lambda_{\min}} \frac{1}{\lambda_{\min} - \lambda}(A - \lambda I)v \\
&= \prod_{\lambda \neq \lambda_{\min}} \frac{1}{\lambda_{\min} - \lambda}(Av - \lambda v) \\
&= \prod_{\lambda \neq \lambda_{\min}} \frac{1}{\lambda_{\min} - \lambda}(\lambda_{\min} v - \lambda v) \\
&= \prod_{\lambda \neq \lambda_{\min}} \frac{1}{\lambda_{\min} - \lambda} \cdot (\lambda_{\min} - \lambda)v \\
&= v.
\end{aligned}
$$

Thus, Z acts as the identity operator when restricted to the eigenspace of $\lambda_{\min}$. So $\mathrm{Im}(Z)$ is the eigenspace of $\lambda_{\min}$ and Z is idempotent. It follows that $Z = E_{\lambda_{\min}}$.

Now as G is 1-walk regular and $E_{\lambda_{\min}}$ is a polynomial in A, there exist constants a, b such that: $E_{\lambda_{\min}} \circ I = aI$ and $E_{\lambda_{\min}} \circ A = bA$. As $E = PP^T$, it follows that: $\langle p_i, p_i \rangle = a$ for all $i \in [n]$, and $\langle p_i, p_j \rangle = b$ for all $i \sim j$.

As $E_{\lambda_{\min}}$ is the projector onto $\ker(A - \lambda_{\min} I)$ and $d = \mathrm{corank}(A - \lambda_{\min} I)$, $\mathrm{tr}(E_{\lambda_{\min}}) = d$. However, as $E_{\lambda_{\min}} \circ I = aI$, $\mathrm{tr}(E_{\lambda_{\min}}) = na$. Thus, $a = d/n$.

Now denote $\mathrm{sum}(M)$ as the sum of the entries in the matrix M. As G is 1-walk regular, G is r-regular for some $r \in \mathbb{N}$. Now as $E_{\lambda_{\min}} \circ A = bA$, it follows that:

$$brn = \mathrm{tr}(A \circ E_{\lambda_{\min}}) = \mathrm{tr}(\lambda_{\min} E_{\lambda_{\min}}) = \lambda_{\min} d.$$

So $b = \dfrac{\lambda_{\min} d}{nr} < 0$. As $\lambda_{\min} < 0$, $b < 0$. Thus, the map $\varphi : V(G) \to \mathbb{R}^d$ sending $\varphi(i) = \sqrt{\frac{n}{d}} p_i$ satisfies:

$$\langle \varphi(i), \varphi(i) \rangle = 1 \text{ for all } i \in V(G) \text{ and } \langle \varphi(i), \varphi(j) \rangle = \frac{\lambda_{\min}}{r} \text{ for all } i \sim j.$$

So φ is a strict vector coloring of G. $\square$

In order to characterize 1-walk regular graphs that are uniquely vector colorable, the notions of a *tensegrity graph* and *tensegrity framework* will be introduced. Intuitively, a tensegrity framework provides a combinatorial abstraction of a physical system, capturing notions of rigidity and flexibility.

Definition 3.3 (Tensegrity Graph)**.** A *tensegrity graph* is defined as a graph $G(V, E)$ where the edge set E is partitioned into three disjoint sets B, C, and S. The elements of B, C, and S are referred to as *bars, cables,* and *struts* respectively.

Definition 3.4 (Tensegrity Framework)**.** A *tensegrity framework* $\mathrm{G}(\mathbf{p})$ consists of a tensegrity graph G, and an assignment of real-valued vectors $\mathbf{p} = (p_1, \ldots, p_n)$ to the vertices of G. Here, if each $p_i \in \mathbb{R}^d$, we denote $G(\mathbf{p}) \subset \mathbb{R}^d$. The associated *framework matrix* is the $n \times d$ matrix, where the vector p_i is the ith row of the matrix.

When working with tensegrity frameworks, the bars, cables, and struts each have some distance (non)-preserving property. The bars preserve the distance of the respective vertices. The cables provide an upper bound on the distance for certain pairs of vertices, and the struts provide a lower bound on the distance for certain pairs of vertices. [13] This perspective provides a means of comparing two tesnegrity frameworks. Suppose $G(\mathbf{p})$ and $G(\mathbf{q})$ are tensegrity frameworks. Intuitively, $G(\mathbf{p})$ improves upon $G(\mathbf{q})$ if the struts provide a greater lower bound and the cables provide a smaller upper bound on the distance for certain pairs of vertices. The distances provided by the bars should remains the same for both $G(\mathbf{p})$ and $G(\mathbf{q})$, if $G(\mathbf{p})$ improves upon $G(\mathbf{q})$. The next definition serves to formalize this notion of improvement.

Definition 3.5. Let $G(\mathbf{p})$ and $G(\mathbf{g})$ be tensegrity frameworks with respect to the graph G. We say that $G(\mathbf{p})$ *dominates* $G(\mathbf{q})$, denoted $G(\mathbf{p}) \succeq G(\mathbf{q})$, if the following three conditions hold:

1. $\langle q_i, q_j \rangle = \langle p_i, p_j \rangle$ for all $ij \in B$ or $i = j$;
2. $\langle q_i, q_j \rangle \geq \langle p_i, p_j \rangle$ for all $ij \in C$;
3. $\langle q_i, q_j \rangle \leq \langle p_i, p_j \rangle$ for all $ij \in S$.

$G(\mathbf{p})$ and $G(\mathbf{q})$ are said to be *congruent* if $\mathrm{Gram}(p_1, \ldots, p_n) = \mathrm{Gram}(q_1, \ldots, q_n)$.

The domination relation is the key tool in characterizing optimal vector colorings. It will be shown that for tensegrity frameworks where all the edges are struts, $G(\mathbf{p}) \succeq G(\mathbf{q})$ if and only if $\mathbf{q}$ is a better vector coloring than $\mathbf{p}$.

Definition 3.6. Let $G(V, E)$ be a graph, and let $\mathbf{p}$ be a vector coloring of G. Denote $\tilde{G}$ be the tensegrity graph by setting $S = E$ and $B = C = \emptyset$, and let $\tilde{G}(\mathbf{p})$ be the corresponding tensegrity framework.

Lemma 3.7. *Let $G(V, E)$ be a graph, and let $\boldsymbol{p}$ be a strict vector coloring of G. The vector coloring $\boldsymbol{q}$ achieves a smaller or equal value compared to the vector coloring $\boldsymbol{p}$ if and only if $\tilde{G}(\boldsymbol{p}) \succeq \tilde{G}(\boldsymbol{q})$.*

Proof. Let t be the value of $\mathbf{p}$ as a strict vector coloring. Suppose $\mathbf{q}$ is a vector coloring r-coloring, for $r \leq t$. So for all $i \sim j$, it follows that:

$$\langle q_i, q_j \rangle \leq \frac{-1}{r-1} \leq \frac{-1}{t-1} = \langle p_i, p_h \rangle.$$

As $B = C = \emptyset$, $\tilde{G}(\mathbf{p}) \succeq \tilde{G}(\mathbf{q})$. Conversely, suppose $\tilde{G}(\mathbf{p}) \succeq \tilde{G}(\mathbf{q})$. As $B = C = \emptyset$ and $S = E$, it follows that:

$$\langle q_i, q_j \rangle \leq \langle p_i, p_j \rangle \text{ for all } i \sim j.$$

So $\mathbf{q}$ achieves a smaller or equal value compared to the vector coloring $\mathbf{p}$. □

Definition 3.8 (Spherical Stress Matrix)**.** Let $G(\mathbf{p}) \subset \mathbb{R}^d$ be a tensegrity framework, and let P be the corresponding framework matrix. A *spherical stress matrix* for $G(\mathbf{p})$ is a real, symmetric $n \times n$ matrix Z with the following properties:

1. Z is positive semidefinite.

2. $Z_{ij} = 0$ whenever $i \neq j$ and $ij \notin E$.

3. $Z_{ij} \geq 0$ for all (struts) $ij \in S$, and $Z_{ij} \leq 0$ for all (cables) $ij \in C$.

4. $ZP = 0$.

5. $\text{corank}(Z) = \dim \text{span}(p_1, \ldots, p_n)$.

We prove a couple technical results, which will be useful in characterizing 1-walk regular graphs that are uniquely colorable.

Lemma 3.9. *Let G be a tensegrity framework with no cables (i.e., $C = \emptyset$), and let A be the adjacency matrix of G. Let $\tau := \lambda_{\min}(A)$. The matrix $A - \tau I$ is a spherical stress matrix for any generalized least eigenvalue framework P of G.*

Proof. As $\tau < 0$, the eigenvalues of $A - \tau I$ are all non-negative. So $A - \tau I$ is positive semidefinite. As A is the adjacency matrix and τI is a scalar of the identity matrix, $(A - \tau I)_{ij} = 0$ whenever $i \neq j$ and $ij \notin E(G)$. As G has no cables, $Z_{ij} \leq 0$ trivially holds whenever $ij \in C$. As $(A - \tau I)_{ij} \geq 0$ whenever $i \neq j$, $Z_{ij} \geq 0$ for all $ij \in S$. Now as the columns of P are eigenvectors of τ, $(A - \tau I)P = 0$. Finally, we note that $\text{corank}(A - \tau I)$ is equal to the dimension of the eigenspace corresponding to τ. This is precisely $\dim \text{span}(p_1, \ldots, p_n)$. $\square$

Lemma 3.10. *Let $X \in \mathcal{S}^n_+$, and let $Y \in \mathcal{S}^n$ satisfy $\ker(X) \subset \ker(Y)$. If $X = PP^T$ for some matrix P, then there exists $R \in \mathcal{S}$ such that:*

$$Y = PRP^T \text{ and } Im(R) \subset Im(P^T).$$

Proof. Suppose first that P has full column rank. We extend P to a full rank, square symmetric matrix Q. Define the matrix $R' := Q^{-1}Y(Q^{-1})^T$. As $\ker(X) \subset \ker(Y)$, it follows that:

$$\ker(X) \oplus 0 = \ker(Q(I \oplus 0)Q^T) \subset \ker(QR'Q^T) = \ker(Y) \oplus 0.$$

As Q is invertible, it follows that $\ker(I \oplus 0) \subset \ker(R')$. Thus, $R' = R \oplus 0$, for some real symmetric matrix R. By construction, we have that $Y = PRP^T$. As P is full rank, $\text{Im}(R) \subset \text{Im}(P^T)$.

Now suppose instead that P does not have full column rank. As X is symmetric and positive-semidefinite, there exists a matrix B with full column rank such that

$X = BB^T$. By the previous case, there exists a symmetric matrix R' such that $Y = BR'B^T$ and $\mathrm{Im}(R') \subset \mathrm{Im}(P^T)$. Now observe that $\mathrm{Im}(X) = \mathrm{Im}(B) = \mathrm{Im}(P)$. Thus, there exists a matrix U such that $B = PU$. Therefore, $Y = (PU)R'(PU)^T$. Now let E be the orthogonal projector onto $\mathrm{Im}(P^T)$. So E is symmetric, $EP^T = P^T$, and $PE = P$. Thus:

$$Y = PEUR'U^TEP^T.$$

Take $R = EUR'U^TE$. So $\mathrm{Im}(R) \subset \mathrm{Im}(E) = \mathrm{Im}(P^T)$. This completes the proof.

□

Theorem 3.11. *Let $G(\boldsymbol{p}) \subset \mathbb{R}^d$ be a tensegrity framework, and let $P \in \mathbb{R}^{n\times d}$ be the corresponding framework matrix. Let $Z \in \mathcal{S}^n_+$ be a spherical stress matrix for $G(\boldsymbol{p})$. The framework $G(\boldsymbol{p})$ dominates the framework $G(\boldsymbol{q})$ if and only if:*

$$\mathit{Gram}(q_1, \ldots, q_n) = PP^T + PRP^T$$

where R is a symmetric $d \times d$ matrix satisfying:

(a) $\mathit{Im}(R) \subset \mathit{span}(p_1, \ldots, p_n)$;

(b) $p_i^T Rp_j = 0$ for $i = j$ and $ij \in B \cup \{\ell k \in C \cup S : Z_{\ell k} \neq 0\}$;

(c) $p_i^T Rp_j \geq 0$ for $ij \in C$;

(d) $p_i^T Rp_j \leq 0$ for $ij \in S$.

Proof. Suppose first there exists a matrix $R \in \mathcal{S}^d$ satisfying (a)-(d), and that:

$$\mathrm{Gram}(q_1, \ldots, q_n) = PP^T + PRP^T.$$

Note that the ij entry of $\mathrm{Gram}(q_1, \ldots, q_n)$ is $\langle q_i, q_j\rangle$, while the ij entry of $PP^T + PRP^T$ is $\langle p_i, p_j\rangle + \langle p_i, Rp_j\rangle$. Thus:

$$\langle q_i, q_j\rangle = \langle p_i, p_j\rangle + \langle p_i, Rp_j\rangle \text{ for all } i, j \in [n].$$

By (b), it follows that $\langle q_i, q_j \rangle = \langle p_i, p_j \rangle$ for all $ij \in B$ or $i = j$. By (c), we have that $\langle q_i, q_j \rangle \geq \langle p_i, p_j \rangle$ for all $ij \in C$. Finally, by (d), it follows that $\langle q_i, q_j \rangle \leq \langle p_i, p_j \rangle$ for all $ij \in S$. Thus, $G(\mathbf{p}) \succeq G(\mathbf{q})$.

Conversely, suppose that $G(\mathbf{p}) \succeq G(\mathbf{q})$. Define $X := PP^T$, which we note is just $\text{Gram}(p_1, \ldots, p_n)$. Similarly, define $Y := \text{Gram}(q_1, \ldots, q_n)$. As Z is a spherical stress matrix for $G(\mathbf{p})$, it follows that $ZX = 0$. So $\text{Im}(X) \subset \ker(Z)$. Using again the fact that Z is a spherical stress matrix for $G(\mathbf{p})$, we have that $\text{corank}(Z) = \text{rank}(X)$. Thus, $\text{Im}(X) = \ker(Z)$. As Y and Z are positive semidefinite and $G(\mathbf{p}) \succeq G(\mathbf{q})$, it follows that:

$$0 \leq \text{tr}(ZY) = \sum_{i \simeq j} Z_{ij} Y_{ij} \leq \sum_{i \simeq j} Z_{ij} X_{ij} = \text{tr}(ZX) = 0. \tag{3.2}$$

So $\text{tr}(ZY) = 0$. We again use the fact that Y and Z are positive semidefinite to obtain that if $\text{tr}(YZ) = 0$, then $YZ = 0$. So $\ker(Y) \supset \text{Im}(Z) = \ker(X)$. It follows that $\ker(X) \subset \ker(Y - X)$. We apply Lemma 3.10 to X and $Y - X$ to obtain that there exists $R \in \mathcal{S}$ such that $Y = PRP^T$ and $\text{Im}(R) \subset \text{Im}(P^T) = \text{span}(p_1, \ldots, p_n)$. So we have that: $\text{Gram}(q_1, \ldots, q_n) = PP^T + PRP^T$.

It will now be shown that R satisfies (a)-(d). By assumption, $G(\mathbf{p}) \succeq G(\mathbf{q})$. So for all $ij \in B$ or $i = j$, it follows that $\langle q_i, q_j \rangle = \langle p_i, p_j \rangle$. So $p_i^T R p_j = 0$ in this case. Similarly, $p_i^T R p_j \leq 0$ for all $ij \in S$, and $p_i^T R p_j \geq 0$ for all $ij \in C$. By (3.2), we have that:

$$\sum_{i \simeq j} Z_{ij}(X_{ij} - Y_{ij}) = 0.$$

As $Z_{lk}(X_{lk} - Y_{lk}) \geq 0$ for all $lk \in C \cup S$, we have that $X_{lk} = Y_{lk}$ for all $lk \in C \cup S$, with $Z_{lk} \neq 0$. $\square$

In [6], the authors characterized the optimal vector colorings for 1-walk regular graphs. This characterization will next be introduced.

Theorem 3.12. *Let $G(V, E)$ be a 1-walk regular graph of order n. Let $G(\boldsymbol{p}) \subset \mathbb{R}^d$ be the least eigenvalue framework, and let $P \in \mathbb{R}^{n \times d}$ be the corresponding framework* *matrix. The vector coloring* $\boldsymbol{q}$ *is optimal if and only if there exists* $R \in \mathcal{S}^d$ *such that:*

$$\mathit{Gram}(q_1, \ldots, q_n) = \frac{n}{d}(PP^T + PRP^T).$$

and $p_i^T R p_j = 0$ *for all* $i = j$ *and* $i \sim j$*.*

Proof. Let $\tilde{G}$ be the tensegrity graph obtained from G, by setting all the edges as struts. Let $\tilde{\mathbf{p}}$ be the vector coloring mapping vertex $i \mapsto \sqrt{\frac{n}{d}} p_i$, where p_i is the ith row vector in P. By Lemma 3.7, $\mathbf{q}$ acheives a smaller value than $\tilde{\mathbf{p}}$ if and only if $\tilde{G}(\tilde{\mathbf{p}}) \succeq \tilde{G}(\mathbf{q})$. Now as P is the least eigenvalue framework matrix for G, we have by Lemma 3.9 that $A - \lambda_{\min} I$ is a spherical stress matrix for $\tilde{G}(\tilde{\mathbf{p}})$. By Theorem 3.11, we have that $\tilde{G}(\tilde{\mathbf{p}}) \succeq \tilde{G}(\mathbf{q})$ if and only if:

$$\mathrm{Gram}(q_1, \ldots, q_n) = \frac{n}{d}(PP^T + PRP^T).$$

for some $R \in \mathcal{S}^d$ satisfying $\tilde{p}_i^T R \tilde{p}_j = 0$ for all $i \simeq j$. Thus, for all $i \simeq j$, we have that: $\langle q_i, q_j \rangle = \langle \tilde{p}_i, \tilde{p}_j \rangle$. So the vector coloring $\mathbf{q}$ achieves the same value as the vector coloring $\tilde{\mathbf{p}}$. $\square$

Corollary 3.13. *Let* $G(V, E)$ *be a* 1*-walk regular graph with degree* k*, and let* $G(\boldsymbol{p}) \subset \mathbb{R}^d$ *be its least eigenvalue framework. Let* $P \in \mathbb{R}^{n \times d}$ *be the corresponding framework matrix. Then* $\chi_v(G) = 1 - \frac{k}{\lambda_{\min}}$ *and the vector coloring* $\tilde{\boldsymbol{p}}$ *mapping vertex* $i \mapsto \sqrt{\frac{n}{d}} p_i$*, where* p_i *is the* i*th row vector of* P*, is an optimal strict vector coloring of* G*.*

Proof. By Lemma 3.2, $\tilde{\mathbf{p}}$ is a strict vector coloring of G. It was established in the proof of Theorem 3.11 that no vector coloring of G acheives a better value than $\tilde{\mathbf{p}}$. In the proof of Lemma 3.2, it was shown that the canonical vector coloring is a $1 - \frac{\lambda_{\min}}{k}$ coloring. $\square$

Corollary 3.14. *Let* $G(V, E)$ *be a* 1*-walk regular graph, and let* $G(\boldsymbol{p}) \subset \mathbb{R}^d$ *be its least eigenvalue framework. Let* $P \in \mathbb{R}^{n \times d}$ *be the corresponding framework matrix.* G *is uniquely vector colorable if and only if for any* $R \in \mathcal{S}^d$*, we have that:*

$$p_i^T R p_j = 0 \text{ for all } i \simeq j \implies R = 0.$$

Proof. We note that G is uniquely vector colorable if and only if, for every vector coloring $\mathbf{q}$ of G:

$$\mathrm{Gram}(q_1, \dots, q_n) = \mathrm{Gram}(p_1, \dots, p_n) = \frac{n}{d} P P^T. \tag{3.3}$$

However, by Theorem 3.11, it follows that:

$$\mathrm{Gram}(q_1, \dots, q_n) = \frac{n}{d}(PP^T + PRP^T).$$

So (3.3) is equivalent to the statement that: if $R \in \mathcal{S}^d$ satisfies $p_i^T R p_j = 0$ for all $i \simeq j$, then $R = 0$. □

3.2 Unique Vector Colorings of Kneser Graphs

In this section, it will be shown that the Kneser Graph is uniquely vector colorable. An explicit vector coloring for the Kneser Graph will also be provided. Note that the Kneser Graph is vertex transitive, and therefore 1-walk regular. Observe that for $n < 2k$, $\mathrm{KG}(n, k)$ has no edges. If instead $n = 2k$, $\mathrm{KG}(n, k)$ is a perfect matching on $\binom{2k}{k}$ vertices, matching $S \in \binom{[n]}{k}$ to its complement $[n] \setminus S$. As a perfect matching is bipartite, $\chi_v(\mathrm{KG}(n, k)) = 2$. So we consider the case of $n \geq 2k + 1$. In order to show that $\mathrm{KG}(n, k)$ is uniquely vector colorable, it is necessary to first construct its generalized least-eigenvalue framework matrix. Let P be a real-valued matrix, whose rows are indexed by the vertices of $\mathrm{KG}(n, k)$ and whose columns are indexed by $[n]$. For a subset $S \in \binom{[n]}{k}$ and element $j \in [n]$, define:

$$P_{S,j} = \begin{cases} k - n : & j \in S, \\ k : & j \notin S. \end{cases}$$

While the columns of P are not necessarily orthogonal, they do span the least eigenspace of $\mathrm{KG}(n, k)$ [4]. Thus, the row vectors of P form a generalized least eigenvalue framework of $\mathrm{KG}(n, k)$. So to show that $\mathrm{KG}(n, k)$ is uniquely vector colorable, we have by Corollary 3.14 that it suffices to show that for any $n \times n$ symmetric matrix R:

$$p_i^T R p_j = 0 \text{ for all } i \simeq j \implies R = 0.$$

Now by construction, $P\vec{1} = 0$, where $\vec{1}$ denotes the all-ones vector. It will first be shown that $\text{span}(\{p_i : i \in V(\text{KG}(n,k))\}) = \{\vec{1}\}^\perp$. So if $\text{Im}(R) \subset \text{span}(\vec{1})$, then $p_i^T R p_j = 0$. Thus, it suffices to check only symmetric matrices R such that $\text{Im}(R) \subset \text{span}(\{p_i : i \in V(\text{KG}(n,k))\})$.

We begin with some notation. For $S \in \binom{[n]}{k}$ and $x \in [n]$, denote:

$$1_{S,x} = \begin{cases} 1: & x \in S, \\ 0: & x \notin S. \end{cases}$$

Now for $S \in \binom{[n]}{k}$, define the following subspaces of $\mathbb{R}^n$:

$$P_S := \text{span}\left(\{p_T : T \cap S = \emptyset\} \cup \{\vec{1}\}\right),$$
$$E_S := \text{span}\left(\{e_i : i \notin S\} \cup \{1_S\}\right).$$

Where p_T is the row vector of P corresponding to $T \in \binom{[n]}{k}$, and e_i is the ith standard basis vector.

Lemma 3.15. *Let $n, k \in \mathbb{N}$ with $n \geq 2k+1$, and let $S \in \binom{[n]}{k}$. Then $P_S = E_S$.*

Proof. We first show that $P_S \subset E_S$. Observe that:

$$\vec{1} = 1_S + \sum_{i \in [n] \setminus S} e_i.$$

So $\vec{1} \in E_S$. Now let $p_T \in P_S$. By the definition of p_T, $T \cap S = \emptyset$. Thus, for each $i \in T$, $i \notin S$. So for each $i \in T$, $e_i \in E_S$. It follows that:

$$p_T = \sum_{i \in T} e_i \in E_S.$$

We conclude that $P_S \subset E_S$. Now let $i \in [n] \setminus S$. We show that $e_i \in P_S$. As $n \geq 2k+1$, there exists a subset $U \subset [n]$ with $|U| = k+1, i \in U$, and $U \cap S = \emptyset$. So for any $T \in \binom{U}{k}$, $T \cap S = \emptyset$. Thus, $p_T \in P_S$. Now recall that $\vec{1} \in P_S$. So for each $T \in \binom{U}{k}$, we have:

$$\vec{1} - \frac{1}{k} p_T = \left(\frac{n}{k} - 1\right) 1_T \in P_S.$$

Thus, $1_T \in P_S$. Consider the incidence matrix whose rows are indexed by the members of $\binom{U}{k}$ and whose columns are indexed by $[n]$. We note that the vectors 1_T are the rows of this matrix, where $T \in \binom{U}{k}$. So this matrix is of the form $[M|0]$. In order to show that $e_i \in P_S$ for all $i \in U$, it suffices to show that $M^T M$ has full column rank. Now observe that $(M^T M)_{ij}$ counts the number of subsets of U that contain both i and j. Note that if $i \neq j$, there are $\binom{k-1}{k-2} = k-1$ elements of $\binom{U}{k}$ that contain both i and j. If $i = j$, there are $\binom{k}{k-1} = k$ elements of $\binom{U}{k}$ that contains i. So:

$$(M^T M)_{ij} = \begin{cases} k-1: & i \neq j, \\ k: & i = j. \end{cases}$$

Thus, $M^T M = I + (k-1)J$, which clearly has only positive eigenvalues. So $M^T M$ is invertible. So for all $j \in U$, $e_j \in P_S$. Finally, it will be shown that $1_S \in P_S$. Observe that:

$$1_S = \vec{1} - \sum_{i \in [n] \setminus S} e_i.$$

So $1_S \in P_S$, and we conclude that $E_S \subset P_S$. □

Corollary 3.16. *Let $n, k \in \mathbb{N}$ such that $n \geq 2k+1$, and consider the graph $KG(n,k)$. We have that:*

$$\mathit{span}\left(\{p_T : T \in V(KG(n,k))\}\right) = \mathit{span}(\{\vec{1}\})^{\perp}.$$

Proof. Set $S = \emptyset$. By Lemma 3.15, it follows that:

$$\text{span}\left(\{p_T : T \in V(\text{KG}(n,k))\} \cup \{\vec{1}\}\right) = \mathbb{R}^n.$$

Thus:

$$\mathrm{span}\left(\{p_T : T \in V(\mathrm{KG}(n,k))\}\right) = \mathrm{span}(\{\vec{1}\})^{\perp}.$$

$\square$

Corollary 3.16 will now be employed to prove that $\mathrm{KG}(n,k)$ is uniquely vector colorable, for $n \geq 2k+1$.

Theorem 3.17. *Let $n, k \in \mathbb{N}$ such that $n \geq 2k+1$. The graph $KG(n,k)$ is uniquely vector colorable.*

Proof. As $G = \mathrm{KG}(n,k)$ is vertex transitive, it is 1-walk regular. Let P be the real-valued matrix, whose rows are indexed by the vertices of $\mathrm{KG}(n,k)$ and whose columns are indexed by $[n]$. For a subset $S \in \binom{[n]}{k}$ and element $j \in [n]$, define:

$$P_{S,j} = \begin{cases} k-n : & j \in S, \\ k : & j \notin S. \end{cases}$$

Recall that P is a generalized least eigenvalue framework matrix of G. By Corollary 3.14, it suffices to show that for any matrix $R \in \mathcal{S}^n$, the following condition holds:

$$p_S^T R p_T = 0 \text{ for all } S \simeq T \implies R = 0. \tag{3.4}$$

By Corollary 3.16, the row space of P is $\mathrm{span}(\{\vec{1}\})^{\perp}$. So if $\mathrm{Im}(R) \subset \mathrm{span}(\vec{1})$, condition (3.4) immediately holds. Thus, it suffices to consider the case $\mathrm{Im}(R) \subset \mathrm{span}(\vec{1})^{\perp}$. It follows from condition (3.4) that p_S^T and Rp_T are orthogonal. Furthermore, as $\mathrm{Im}(R) \subset \mathrm{span}(\{\vec{1}\})^{\perp}$, Rp_T is orthogonal to $\vec{1}$. So Rp_T is orthogonal to: P_T, where we recall:

$$P_T := \mathrm{span}\left(\{p_T : T \cap S = \emptyset\} \cup \{\vec{1}\}\right).$$

By Lemma 3.15, $P_T = E_T$. So for $i \notin T$, Rp_T and e_i are orthogonal. As R is symmetric, it follows that p_T is orthogonal to Re_i for all T not containing i. It follows from this, and the fact that $\mathrm{Im}(R) \subset \mathrm{span}(\vec{1})^{\perp}$ to deduce that Re_i is orthogonal to P_F where $F = \{i\}$. By Lemma 3.15, $P_F = E_F$. Observe that $1_F = e_i$. So $E_F = \{e_j : j \in [n]\} = \mathbb{R}^n$. As i was arbitrary, it follows that $R = 0$. $\square$

Remark: The graph $\mathrm{KG}(n,k)$ is $\binom{n-k}{k}$ regular with smallest eigenvalue $\lambda_{min} = -\binom{n-k-1}{k-1}$ [3]. So for $n \geq 2k+1$:

$$\chi_v(\mathrm{KG}(n,k)) = 1 + \frac{\binom{n-k}{k}}{\binom{n-k-1}{k-1}}. \tag{3.5}$$

Each row of the generalized least framework matrix P we constructed has norm $\sqrt{nk(n-k)}$. For $n \geq 2k+1$, we construct an optimal vector coloring for $\mathrm{KG}(n,k)$ as follows. The vertex S of $\mathrm{KG}(n,k)$ is assigned the vector p_S, which is defined as follows:

$$p_S(i) = \begin{cases} \frac{k-n}{\sqrt{nk(n-k)}} : & i \in S, \\ \frac{k}{\sqrt{nk(n-k)}} : & i \notin S. \end{cases}$$

Now let $S, T \in V(\mathrm{KG}(n,k))$, and denote $h := |S \cap T|$. Observe that:

$$\begin{aligned} \langle p_S, p_T \rangle &= \frac{1}{nk(n-k)} \cdot \left(h(k-n)^2 + 2(k-h)k(k-n) + (n+h-2k)k^2\right) \\ &= \frac{1}{nk(n-k)} \cdot \left[h\left((k-n)^2 - 2k(k-n) + k^2\right) + \left((n-2k)k^2 + 2k^2(k-n)\right)\right] \\ &= \frac{1}{nk(n-k)} \cdot \left(hn^2 - k^2 n\right) \\ &= \frac{h}{k} \cdot \frac{n/k}{n/k-1} - \frac{1}{n/k-1}. \end{aligned}$$

In particular, when $h = 0$, we have that:

$$\frac{h}{k} \cdot \frac{n/k}{n/k-1} - \frac{1}{n/k-1} = -\frac{1}{n/k-1}.$$

So $\chi_v(\mathrm{KG}(n,k)) = n/k$. In particular, it follows from (3.5) that:

$$\frac{n}{k} = 1 + \frac{\binom{n-k}{k}}{\binom{n-k-1}{k-1}}.$$

3.3 Unique Vector Colorings of q-Kneser Graphs

It is straight-forward to modify the proof that $\text{KG}(n, k)$ is uniquely vector colorable (for $n \geq 2k + 1$) to show that q-$\text{KG}(n, k)$ is uniquely vector colorable (again, for $n \geq 2k+1$). At a high level, the k-element subsets are replaced by the k-dimensional subspaces of $\mathbb{F}_q^n$, and elements of $[n]$ are replaced by the 1-dimensional subspaces of $\mathbb{F}_q^n$, or lines. Finally, recall that the canonical vector coloring $\mathbf{p}$ of $\text{KG}(n, k)$ utilizes the integers n and k. The canonical vector coloring of $q - \text{KG}(n, k)$ is essentially identical to $\mathbf{p}$, replacing n and k with the quantum integers $[n]_q$ and $[k]_q$.

The standard Kneser graph shares a number of properties with the q-Kneser graph. Both graphs are vertex transitive, and so 1-walk regular. Just as is the case with the Kneser graph, q-$\text{KG}(n, k)$ has no edges when $n < 2k$ and is a perfect matching when $n = 2k$. So we restrict attention to q-$\text{KG}(n, k)$ for configurations $n \geq 2k + 1$ [4]. The least-eigenvalue framework matrix for q-$\text{KG}(n, k)$ will now be constructed. For a k-dimensional subspace $S \in \text{Gr}_n(k, \mathbb{F}_q)$ and line ℓ in $\mathbb{F}_q^n$, define:

$$P_{S,\ell} = \begin{cases} [k]_q - [n]_q : & \ell \subset S, \\ [k]_q : & \ell \not\subset S. \end{cases}$$

In order to show that $\text{KG}(n, k)$ is uniquely vector colorable, we have by Corollary 3.14 that it suffices to show that for any $n \times n$ symmetric matrix R:

$$p_i^T R p_j = 0 \text{ for all } i \simeq j \implies R = 0.$$

Fix $S \in \text{Gr}_n(k, \mathbb{F}_q)$. Now S contains $[k]_q$ lines. So there are $[n]_q - [k]_q$ lines in $\mathbb{F}_q^n$ that are not contained in S. Let p_S be the row indexed by S in P. Observe that:

$$\langle p_S, \vec{1} \rangle = [k]_q([k]_q - [n]_q) + ([n]_q - [k]_q)k_q = 0.$$

Thus, $P\vec{1} = 0$. It will next be shown that $\text{span}(\{p_S : S \in V(q\text{-KG}(n, k))\}) = \text{span}(\vec{1})^{\perp}$. In light of this fact and Corollary 3.14, q-$\text{KG}(n, k)$ is uniquly vector colorable if and only if for any $[n]_q \times [n]_q$ symmetric matrix R satisfying $\text{Im}(R) \subset \text{span}(\vec{1})$:

$$p_i^T R p_j = 0 \text{ for all } i \simeq j \implies R = 0.$$

For $S \in \text{Gr}_n(k, \mathbb{F}_q)$ and the line $\ell \subset \mathbb{F}_q^n$, denote:

$$1_{S,\ell} = \begin{cases} 1: & \ell \subset S, \\ \\ 0: & \ell \not\subset S. \end{cases}$$

Now for $S \in \text{Gr}_n(k, \mathbb{F}_q)$, define the following subspaces of $\mathbb{R}^{[n]_q}$:

$$P_S := \text{span}\left(\{p_T : T \cap S = \{0\}\} \cup \{\vec{1}\}\right),$$
$$E_S := \text{span}\left(\{e_\ell : \ell \not\subset S\} \cup \{1_S\}\right),$$

where p_T is the row vector of P corresponding to $T \in \text{Gr}_n(k, \mathbb{F}_q)$, and e_ℓ is the standard basis vector indexed by the line $\ell \subset \mathbb{F}_q^n$.

Lemma 3.18. *Let $n, k \in \mathbb{N}$ with $n \geq 2k+1$, and let $S \in Gr_n(k, \mathbb{F}_q)$. Then $P_S = E_S$.*

Proof. It will be shown that $P_S \subset E_S$. Observe that:

$$\vec{1} = 1_S + \sum_{\ell \text{ skew to } S} e_\ell.$$

So $\vec{1} \in E_S$. Now let $p_T \in P_S$. As $T \cap S = \{0\}$ by definition of p_T, it follows that $\ell \not\subset S$ for each $\ell \subset T$. So $e_\ell \in E_S$. Thus:

$$p_T = \sum_{\ell \subset T} e_\ell.$$

So $p_T \in E_S$, and we conclude that $P_S \subset E_S$. Now let ℓ be a line skew to S. It will be shown that $e_\ell \in P_S$. As $n \geq 2k+1$, there exists a $U \in \text{Gr}_n(k+1, \mathbb{F}_q)$ with $\ell \subset U$ and $U \cap S = \{0\}$. So for any k-dimensional subspace T of U, we have that $T \cap S = \{0\}$. Thus, $p_T \in P_S$. Now recall that $\vec{1} \in P_S$. So for each k-dimensional subspace T of U, we have:

$$\vec{1} - \frac{1}{k} p_T = \left(\frac{n}{k} - 1\right) 1_T \in P_S.$$

Thus, $1_T \in P_S$. Consider the incidence matrix whose rows are indexed by the k-dimensional subspaces of U and whose columns are indexed by the lines of $\mathbb{F}_q^n$. Observe that the vectors 1_T are the rows of this matrix, where T is a k-dimensional subspace of U. So this matrix is of the form $[M|0]$. In order to show that $e_\ell \in P_S$ for all $\ell \subset U$, it suffices to show that M^TM has full column rank. Now observe that $(M^TM)_{ij}$ counts the number of subspaces of U that contain both i and j. Note that if $i \neq j$, there are $\begin{bmatrix} k-1 \\ k-2 \end{bmatrix}_q = [k-1]_q$ k-dimensional subspaces of U that contain both i and j. If $i = j$, there are $\begin{bmatrix} k \\ k-1 \end{bmatrix}_q = [k]_q$ k-dimensional subspaces of U that contains i. So:

$$(M^TM)_{ij}) = \begin{cases} [k-1]_q : & i \neq j, \\ [k]_q : & i = j. \end{cases}$$

Thus: $M^TM = q^kI + [k-1]_qJ$, which clearly has only positive eigenvalues. So M^TM is invertible. So for all $j \in U$, $e_j \in P_S$. Finally, it will be shown that $1_S \in P_S$. Observe that:

$$1_S = \vec{1} - \sum_{\ell \text{ skew to } S} e_\ell.$$

So $1_S \in P_S$, and we conclude that $E_S \subset P_S$. $\square$

Corollary 3.19. *Let $n, k \in \mathbb{N}$ such that $n \geq 2k+1$, and consider the graph $KG(n,k)$. We have that:*

$$span\left(\{p_T : T \in V(KG(n,k))\}\right) = span(\{\vec{1}\})^\perp.$$

Proof. We set $S = \{0\}$. By Lemma 3.18, we obtain that:

$$\operatorname{span}\left(\{p_T : T \in V(q\text{-KG}(n,k))\} \cup \{\vec{1}\}\right) = \mathbb{R}^{[n]_q}.$$

Thus:

$$\operatorname{span}\left(\{p_T : T \in V(q\text{-KG}(n,k))\}\right) = \operatorname{span}(\{\vec{1}\})^\perp.$$

$\square$

We now employ Corollary 3.19 to prove that q-KG(n,k) is uniquely vector colorable, for $n \geq 2k+1$.

Theorem 3.20. *Let $n, k \in \mathbb{N}$ such that $n \geq 2k+1$. The graph q-KG(n,k) is uniquely vector colorable.*

Proof. As $G = q\text{-KG}(n,k)$ is vertex transitive, it is 1-walk regular. Let P be the real-valued matrix, whose rows are indexed by the vertices of q-KG(n,k) and whose columns are indexed by the lines of $\mathbb{F}_q^n$. For $S \in \mathrm{Gr}_n(k, \mathbb{F}_q)$ and line ℓ in $\mathbb{F}_q^n$, define:

$$P_{S,\ell} = \begin{cases} [k]_q - [n]_q : & j \in S, \\ [k]_q : & j \notin S. \end{cases}$$

Recall that P is a generalized least eigenvalue framework matrix of G. By Corollary 3.14, it suffices to show that for any matrix $R \in \mathcal{S}^{[n]_q}$, we have that:

$$p_S^T R p_T = 0 \text{ for all } S \simeq T \implies R = 0. \tag{3.6}$$

By Corollary 3.19, the row space of P is $\mathrm{span}(\vec{1})^\perp$. So if $\mathrm{Im}(R) \subset \mathrm{span}(\vec{1})$, condition (3.6) immediately holds. Thus, it suffices to consider the case $\mathrm{Im}(R) \subset \mathrm{span}(\vec{1})^\perp$. By condition (3.6), we have that p_S^T and Rp_T are orthogonal. Furthermore, as $\mathrm{Im}(R) \subset \mathrm{span}(\vec{1})^\perp$, it follows that Rp_T is orthogonal to $\vec{1}$. So Rp_T is orthogonal to P_T, where we recall:

$$P_T := \mathrm{span}\left(\{p_T : T \cap S = \emptyset\} \cup \{\vec{1}\}\right).$$

By Lemma 3.18, $P_T = E_T$. So for a line $i \not\subset T$, Rp_T and e_i are orthogonal. As R is symmetric, it follows that p_T is orthogonal to Re_i for all T not containing i. We deduce from this and the fact that $\mathrm{Im}(R) \subset \mathrm{span}(\vec{1})^\perp$, that Re_i is orthogonal to P_F where $F = \{i\}$. By Lemma 3.18, $P_F = E_F$. Observe that $1_F = e_i$. So $E_F = \{e_j : \ell \text{ is a line of } \mathbb{F}_q^n\} = \mathbb{R}^{[n]_q}$. As i was arbitrary, it follows that $R = 0$. $\square$

Remark: The graph q-KG(n,k) is $q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}_q$-regular with smallest eigenvalue $\lambda_{min} = -q^{k(k-1)}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q$ [11]. So for $n \geq 2k+1$:

$$\chi_v(q\text{-KG}(n,k)) = 1 + \frac{q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q}{\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}. \tag{3.7}$$

Each row of the generalized least framework matrix P has norm $\sqrt{[n]_q[k]_q([n]_q - [k]_q)}$. For $n \geq 2k+1$, an optimal vector coloring for q-KG(n,k) is constructed as follows. The vertex S of q-KG(n,k) is assigned the vector p_S, which is defined as follows:

$$p_S(\ell) = \begin{cases} \frac{[k]_q - [n]_q}{\sqrt{[n]_q[k]_q([n]_q-[k]_q)}} : & \ell \subset S, \\ \frac{[k]_q}{\sqrt{[n]_q[k]_q([n]_q-[k]_q)}} : & \ell \not\subset S. \end{cases}$$

Now let $S, T \in V(q\text{-KG}(n,k))$, and denote $h := \dim(S \cap T)$. It follows that:

$$\begin{aligned}
&\langle p_S, p_T \rangle = \\
&\frac{\Big([h]_q([k]_q - [n]_q)^2 + 2([k]_q - [h]_q)[k]_q([k]_q - [n]_q) + ([n]_q + [h]_q - 2[k]_q)[k]_q^2\Big)}{[n]_q[k]_q([n]_q - [k]_q)} \\
&= \frac{[h]_q\Big(([k]_q - [n]_q)^2 - 2[k]_q([k]_q - [n]_q) + [k]_q^2\Big)}{[n]_q[k]_q([n]_q - [k]_q)} \\
&\quad + \frac{\Big(([n]_q - 2[k]_q)[k]_q^2 + 2[k]_q^2([k]_q - [n]_q)\Big)}{[n]_q[k]_q([n]_q - [k]_q)} \\
&= \frac{[h]_q[n]_q^2}{[n]_q[k]_q([n]_q - [k]_q)} - \frac{[k]_q^2[n]_q}{[n]_q[k]_q([n]_q - [k]_q)} \\
&= \frac{[h]_q}{[k]_q} \cdot \frac{[n]_q/[k]_q}{[n]_q/[k]_q - 1} - \frac{1}{[n]_q/[k]_q - 1}.
\end{aligned}$$

In particular, when $h = 0$, we have that:

$$\frac{[h]_q}{[k]_q} \cdot \frac{[n]_q/[k]_q}{[n]_q/[k]_q - 1} - \frac{1}{[n]_q/[k]_q - 1} = -\frac{1}{[n]_q/[k]_q - 1}.$$

So $\chi_v(q\text{-KG}(n,k)) = [n]_q/[k]_q$. In particular, it follows from (3.7) that:

$$\frac{[n]_q}{[k]_q} = 1 + \frac{q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q}{\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}.$$

Fix $k \in [n]$, and consider the k-distance graph of Q_n, which has vertex set $V(Q_n)$. Two vertices u and v are adjacent in the k-distance graph of Q_n if and only if $d(u, v) = k$ in Q_n. Denote $C_{n,k}$ as the set of vectors in $\mathbb{F}_2^n$ with k bits. The elements of $C_{n,k}$ are said to have *weight* k. We note that $Q_n \cong \text{Cay}(\mathbb{F}_2^n, C_{n,1})$. In a similar manner, the k-distance graph of Q_n may be defined as $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$. Observe that when k is odd, $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$ is bipartite. However, when k is even and $k \neq n$, $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$ has two non-bipartite, isomorphic components which correspond to the even and odd weight vectors, respectively. The component of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$ with even weight vectors is referred to as $H_{n,k}$. In this section, it will be shown that $H_{n,k}$ is uniquely vector colorable, for even values of k in the interval $[n/2 + 1, n - 1]$. The main tool is Corollary 3.14. We first show that for even values of k, $H_{n,k}$ is 1-walk regular. To do so, it suffices to show that $H_{n,k}$ is vertex transitive.

Lemma 3.21. *Let $n \in \mathbb{N}$, and let $0 < k \leq n$ be even. Then $H_{n,k}$ is vertex transitive.*

Proof. Denote $\Gamma := \{v \in \mathbb{F}_2^n : \sum_{i=0}^{n} v_i \equiv 0 \pmod 2\} = V(H_{n,k})$. Observe that Γ is closed under the inherent addition operation from $\mathbb{F}_2^n$. So Γ forms a subgroup of $\mathbb{F}_2^n$. The natural left action of Γ on itself induces a transitive action on the vertices of $H_{n,k}$. In particular, for any $u, v \in V(H_{n,k})$, the element $u + v$ in Γ maps u to v. So $H_{n,k}$ is vertex-transitive; and thus, 1-walk regular. $\square$

Now observe that $H_{n,k}$ is a $\binom{n}{k}$-regular graph. In order to determine $\chi_v(H_{n,k})$ for $k \in [n/2+1, n-1]$, it is necessary to determine the minimum eigenvalue $\lambda_{\min}$ of $H_{n,k}$. Denote $A(H_{n,k})$ as the adjacency matrix of $H_{n,k}$. As $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$ consists of two isomorphic components, the adjacency matrix of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$, which is denoted A, can be written as:

$$A = \begin{pmatrix} A(H_{n,k}) & O \\ O & A(H_{n,k}) \end{pmatrix}.$$

So the smallest eigenvalue of $H_{n,k}$ is precisely the smallest eigenvalue of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$. We appeal to a result of Babai [1], which classifies the eigenvalues and eigenvectors of Cayley graphs over abelian groups. Let Γ be an abelian group, and let $C \subset \Gamma$ such that C is closed under inverses and $0 \notin C$. The eigenvalues and eigenvectors of $\text{Cay}(\Gamma, C)$ are determined exactly by the characters of Γ. Precisely, if χ is a character of Γ, then $\sum_{c \in C} \chi(c)$ is an eigenvalue of $\text{Cay}(\Gamma, C)$ with the corresponding eigenvector $v = (\chi(g))_{g \in \Gamma}$. In particular, there are $|\Gamma|$ characters for Γ, which provides a full orthogonal eigenbasis.

This result will be leveraged to ascertain the smallest eigenvalue of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$. The characters of $\mathbb{F}_2^n$ are of the form $\chi_a(x) = (-1)^{a \cdot x}$, where $a \in \mathbb{F}_2^n$ is fixed and $a \cdot x$ is the dot product considered over $\mathbb{F}_2^n$. Denote $a^{\perp} = \{y \in \mathbb{F}_2^n : a \cdot y = 0\}$. Observe that $\chi_a(x) = 1$ if and only if $x \in a^{\perp}$. So the eigenvalue corresponding to χ_a is given by:

$$\lambda_a = \sum_{c \in C_{n,k}} (-1)^{a \cdot c} = |C_{n,k} \cap a^{\perp}| - |C_{n,k} \setminus a^{\perp}| = \binom{n}{k} - 2|C_{n,k} \setminus a^{\perp}|. \tag{3.8}$$

So to determine the smallest eigenvalue of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$, it suffices to determine the elements $a \in \mathbb{F}_2^n$ maximize $|C_{n,k} \setminus a^{\perp}|$. We cite the bound from [6], that for all $a \in \mathbb{F}_2^n$: $|C_{n,k} \setminus a^{\perp}| \leq \binom{n-1}{k-1}$, with equality if and only if a has weight 1 or weight $n-1$. Thus, we have that:

$$\lambda_{\min}(\text{Cay}(\mathbb{F}_2^n, C_{n,k})) = \frac{n-2k}{k}\binom{n-1}{k-1},$$

with multiplicity $2n$. Corollary 3.14 will now be employed to determine $\chi_v(H_{n,k})$ and construct the optimal (strict) vector coloring of $H_{n,k}$.

Theorem 3.22. *Let $k \in [n/2+1, n-1]$ be an even integer. The graph $H_{n,k}$ is uniquely vector colorable, with vector chromatic number:*

$$\chi_v(H_{n,k}) = \frac{2k}{2k-n}.$$

Furthermore, the canonical vector coloring of $H_{n,k}$ *is given by* $u \mapsto p_u \in \mathbb{R}^n$, *where:*

$$p_u(j) = \frac{(-1)^{u_j}}{\sqrt{n}} \text{ for all } j \in [n].$$

Proof. By Lemma 3.21, $H_{n,k}$ is vertex transitive; and therefore, 1-walk regular. So by Corollary 3.13, we have that:

$$\chi_v(H_{n,k}) = 1 - \frac{\deg(H_{n,k})}{\lambda_{\min}} = \frac{2k}{2k-n}.$$

Let P denote the least eigenvalue framework matrix of $H_{n,k}$. Corollary 3.13 also provides that the normalized row vectors of P form an optimal strict vector coloring of $H_{n,k}$. The matrix P will be explicitly constructed. Recall that:

$$\lambda_{\min}(H_{n,k}) = \lambda_{\min}(\text{Cay}(\mathbb{F}_2^n, C_{n,k})) = \frac{n-2k}{k}\binom{n-1}{k-1}.$$

Note that $\lambda_{\min}(\text{Cay}(\mathbb{F}_2^n, C_{n,k}))$ has multiplicity $2n$. Recall the adjacency matrix A of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$ can be written as:

$$A = \begin{pmatrix} A(H_{n,k}) & O \\ O & A(H_{n,k}) \end{pmatrix}.$$

So $\lambda_{\min}(H_{n,k})$ has multiplicity n. Denote v_a as the eigenvector of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$ corresponding to the character χ_a. So the set of orthogonal eigenvectors of $\lambda_{\min}$ of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$ corresponding to $\lambda_{\min}$ are: $\{v_{e_i}\}_{i=1}^n \cup \{v_{\vec{1}+e_i}\}_{i=1}^n$. Now for any eigenvector v_a of $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$, we may write $v_a = (x_a, y_a)$, where x_a is the restriction of v_a to $H_{n,k}$, and y_a is the restriction of v_a to $\mathbb{F}_2^n \setminus V(H_{n,k})$. Furthermore, observe that for each $a \in \mathbb{F}_2^n$, the following hold:

$$v_a(z) = v_{\vec{1}+a}(z) \text{ for all } z \in V(H_{n,k}),$$

$$v_a(z) = -v_{\vec{1}+a}(z) \text{ for all } z \in \mathbb{F}_2^n \setminus V(H_{n,k}).$$

It follows that for all $i \in [n]$, if $v_{e_i} = (x_i, y_i)$ then $v_{\vec{1}+e_i} = (x_i, -y_i)$. It will now be shown that the vectors $\{x_i\}_{i=1}^n$ span $\mathbb{R}^n$. As the eigenvectors in $\{v_{e_i}\}_{i=1}^n \cup \{v_{\vec{1}+e_i}\}_{i=1}^n$

are pairwise orthogonal, we have for all distinct $i, j \in [n]$ that:

$$\langle v_{e_i}, v_{e_j} \rangle = \langle v_{\vec{1}+e_i}, v_{\vec{1}+e_j} \rangle = 0.$$

So the vectors $\{x_i\}_{i=1}^n$ are orthogonal. Thus: $\left\{ \frac{x_i}{\sqrt{2^{n-1}}} : i \in [n] \right\}$ is an orthonormal basis for the eigenspace of $H_{n,k}$ corresponding to $\lambda_{\min}$. So the matrix P, whose ith column is given by $\frac{x_i}{\sqrt{2^{n-1}}}$, is the generalized least eigenvalue framework matrix of $H_{n,k}$. The canonical vector coloring of $H_{n,k}$ is obtained by scaling the rows of P by $\sqrt{\frac{2^{n-1}}{n}}$. As x_i is the restriction of v_{e_i} to $H_{n,k}$, it follows that:

$$p_u(j) = \frac{(-1)^{u_j}}{\sqrt{n}} \text{ for all } j \in [n].$$

This completes the proof. □

Theorem 3.23. *Let $k \in [n/2+1, n]$ be an even integer. The graph $H_{n,k}$ is uniquely vector colorable.*

Proof. By Lemma 3.21, we have that $H_{n,k}$ is vertex transitive, and therefore 1-walk regular. Let P be the least eigenvalue framework matrix of $H_{n,k}$, which was constructed in Theorem 3.22. So by Corollary 3.14, $H_{n,k}$ is uniquely vector colorable if and only if for all matrices $R \in \mathcal{S}^n$:

$$p_i^T R p_j = 0 \text{ for all } i \simeq j \implies R = 0. \tag{3.9}$$

As $\{p_x : x \in V(H_{n,k})\}$ spans $\mathbb{R}^n$, it suffices to show that $Rp_x = 0$ for all $x \in V(H_{n,k})$. Let $x \in V(H_{n,k})$, and consider the subspace:

$$V_x = \text{span}(\{p_y : y \in V(H_{n,k}), y \simeq x\}).$$

As R is symmetric, condition (3.9) is equivalent to $Rp_x \in V_x^\perp$ for all $x \in V(H_{n,k})$. We note that if $V_x = \mathbb{R}^n$ for all $x \in V(H_{n,k})$, then $Rp_x = 0$ for all $x \in V(H_{n,k})$.

It will first be shown that $V_0 = \mathbb{R}^n$, where V_0 is the subspace V_x for $x = 0 \in \mathbb{F}_2^n$. Note that the neighbors of 0 in $H_{n,k}$ are the vectors of weight k in $\mathbb{F}_2^n$. Let $i, j \in [n]$

be distinct, and let $y, z \in C_{n,k}$ such that y and z differ only in positions i and j. So $e_i - e_j = \frac{\sqrt{n}}{2}(p_y - p_z) \in V_0$. Observe that:

$$\text{span}(\{e_i - e_j : i, j \in [n], i \neq j\}) = \text{span}(\vec{1})^{\perp}.$$

Recall that $p_0 = \frac{1}{\sqrt{n}}\vec{1}$, so $\vec{1} \in V_0$. Thus, $V_0 = \mathbb{R}^n$.

Now let $x \in \mathbb{F}_2^n$, and let $\text{Diag}(p_x)$ be the diagonal $n \times n$ matrix, whose diagonal entries are the elements of p_x. Now let $y \simeq 0$, so $p_y \in V_0$. For each $i \in [n]$, we have that:

$$\text{Diag}(p_x)p_y(i) = \begin{cases} -\frac{1}{n} & : p_x(i) \neq p_y(i), \\ \frac{1}{n} & : p_x(i) = p_y(i). \end{cases}$$

So $\text{Diag}(p_x)p_y = p_{x+y}$, where $x+y$ is considered in $\mathbb{F}_2^n$. Observe that the natural left action of $\mathbb{F}_2^n$ on itself induces a faithful action on $\text{Cay}(\mathbb{F}_2^n, C_{n,k})$. Thus, $N(x) = x \cdot N(0)$, where:

$$x \cdot N(0) := \{x + y : y \in N(0)\}.$$

Thus, $\text{Diag}(p_x)V_0 \subset V_x \subset \mathbb{R}^n$. As $\text{Diag}(p_x)$ has full rank and $V_0 = \mathbb{R}^n$, it follows that $V_x = \mathbb{R}^n$. As x was arbitrary, it follows that $V_x = \mathbb{R}^n$ for all $x \in \mathbb{F}_2^n$. The result follows. $\square$

Chapter 4

Cores and Homomorphisms From Specific Graph Classes

4.1 Cores and Vector Colorings

The goal of this section is to establish a sufficient condition for a connected graph to be a core. The notion of local injectivity will be leveraged. Informally, a graph homomorphism is locally injective if it is injective on the neighborhood of each vertex. Let H be a fixed graph. If a graph G is connected and every graph homomorphism $\varphi : V(G) \to V(H)$ is locally injective, then G is a core. Recall that vector t-colorings are graph homomorphisms into $\mathcal{S}_t^d$ (for some d). Using this fact, a relation between vector colorings and cores is established.

Definition 4.1. Let G and H be graphs, and let $\varphi : V(G) \to V(H)$ be a graph homomorphism. We say that φ is *locally injective* if for any $u, v \in V(G)$ that share a common neighbor, $\varphi(u) \neq \varphi(v)$.

Recall that a graph is a core if every endomorphism is an automorphism. The following result of Nešetřil [12] serves as the basis to establish a connection between local injectivity and cores.

Theorem 4.2. *Let G be a connected graph. Every locally injective endomorphism of G is an automorphism.*

Proof. The proof is by induction on $|G|$. When $|G| = 2$, $G \cong K_2$. The only locally injective endomorphisms of G are the identity map, and the map exchanging the vertices of G. These endomorphisms are the only automorphisms of G. Now fix $|G| > 2$, and suppose that the theorem statement holds for every graph H with $|H| < |G|$. Suppose to the contrary that there exists a locally injective $\varphi \in \text{End}(G) \setminus \text{Aut}(G)$. So there exists a vertex $v \in V(G) \setminus \varphi(V(G))$. Let $G_1, \ldots, G_k$ be the components of $V(G) \setminus \{a\}$. Now for each $i \in [k]$, φ restricted to G_i is a locally injective endomorphism. So by the inductive hypothesis, φ restricted to G_i is an automorphism of G_i. Now as $v \notin \varphi(V(G))$, $\varphi(v) \in G_i$ for some $i \in [k]$. As G is connected, $\varphi(G)$ is connected. So $\varphi(V(G)) \subset G_i$. So for every $u \in N(v)$, we have by the Pigeonhole Principle that $\varphi(N(u)) \neq N(\varphi(u))$. So φ is not locally injective, a contradiction. □

Lemma 4.3. *A connected graph G is a core if and only if there is a (possibly infinite) graph H such that $Hom(G, H) \neq \emptyset$ and every $\varphi \in Hom(G, H)$ is locally injective.*

Proof. If G is a core, then every endomorphism of G is an automorphism. So we take $H = G$ and are done. Conversely, suppose that G is connected and not a core. As G is not a core, there exists $\tau \in \text{End}(G) \setminus \text{Aut}(G)$. By Theorem 4.2, τ is not locally injective. So for any $\rho \in \text{Hom}(G, H)$, $\rho \circ \tau$ is not locally injective. □

We apply the above lemma, using the graph $H = \mathcal{S}_t^d$ for $d \in \mathbb{N}$ and $t \geq 2$, to relate vector colorings and cores.

Theorem 4.4. *Let G be a connected graph. If every optimal vector coloring is locally injective, then G is a core.*

Proof. Let $t := \chi_v(G)$, and let:

$$d := \max\{k | \rho : V(G) \to \mathcal{S}_t^k \text{ is an optimal vector coloring}\}.$$

Let $H := \mathcal{S}_t^d$. By construction, $\text{Hom}(G, H)$ is precisely the set of optimal vector colorings of G. By assumption, every optimal vector coloring of G is locally injective. So we apply Lemma 4.3 to deduce that G is a core. □

We apply Theorem 4.4 in the following manner to obtain several families of cores. Suppose G is connected and uniquely vector colorable. If the unique vector coloring of G is locally injective, then by Theorem 4.4, G is a core.

Corollary 4.5. *For $n \geq 2k+1$, $KG(n,k)$ and q-$KG(n,k)$ are cores.*

Proof. By Theorem 3.17, $\mathrm{KG}(n,k)$ is uniquely vector colorable. Similarly, by Theorem 3.20 q-KG(n,k) is uniquely vector colorable. The vector colorings of $\mathrm{KG}(n,k)$ and q-$\mathrm{KG}(n,k)$ constructed at the end of Sections 3.3 and 3.4, respectively, are injective. So by Theorem 4.4 $\mathrm{KG}(n,k)$ and q-$\mathrm{KG}(n,k)$ are cores. □

Corollary 4.6. *Let $k \in [n/2+1, n-1]$, the graph $H_{n,k}$ is a core.*

Proof. By Theorem 3.23, $H_{n,k}$ is uniquely vector colorable. The canonical vector coloring of $H_{n,k}$ constructed in Theorem 3.22 is injective. So by Theorem 4.4, $H_{n,k}$ is a core. □

4.2 Homomorphisms Between Kneser and q-Kneser Graphs

Let G and H be Kneser graphs. While the full set $\mathrm{Hom}(G,H)$ is unknown, an extensive list of homomorphisms between G and H is known. One noteable result is the following due to Stahl [14]: there exists a homomorphism $\varphi : \mathrm{KG}(n,k) \to \mathrm{KG}(n',k')$ if and only if n' is an integer multiple of n, in which case k' is an integer multiple of k as well. Stahl's proof employed the Erdós-Ko-Rado Theorem. In this section, an alternative proof of this result due to [5] is provided. The following lemma is the primary tool. Lemma 4.7 also provides for an analogous necessary condition for the existence of a homomorphism between q-Kneser graphs.

Lemma 4.7. *Let G and H be graphs where G is uniquely vector colorable and $\chi_v(G) = \chi_v(H)$. Let M be the Gram matrix of an optimal vector coloring of H.*

If $\varphi : V(G) \to V(H)$ is a homomorphism, the principal submatrix of M corresponding to $\{\varphi(g)\}_{g \in V(G)}$ is the Gram matrix of the unique optimal vector coloring of G.

Proof. Let ψ be an optimal vector coloring of H. By Lemma 2.65, $\psi \circ \varphi$ is an optimal vector coloring of G. In particular, as G is uniquely vector colorable, $\psi \circ \varphi$ is the unique vector coloring of G. So $\{\varphi(g)\}_{g \in V(G)}$ is the Gram matrix of the unique optimal vector coloring of G. $\square$

Theorem 4.8. *Let $n, n', k, k' \in \mathbb{Z}^+$ satisfying $n > 2k$ and $n/k = n'/k'$. Then there exists a homomorphism from KG(n,k) to KG(n',k') if and only if n' is a multiple of n and k' is a multiple of k.*

Proof. Suppose first that $n' = nq$ and $k' = kq$. We view the vertices of $\mathrm{KG}(n', k')$ as subsets of size k', drawn from the set $[n] \times [q]$. Let $\varphi : V(\mathrm{KG}(n,k)) \to V(\mathrm{KG}(n',k'))$ be given by mapping the vertex S in $\mathrm{KG}(n,k)$ to $S \mapsto [q] \times S$, which is a vertex in $\mathrm{KG}(n',k')$. Now suppose that S and T are adjacent vertices in $\mathrm{KG}(n,k)$. So $S \cap T = \emptyset$. By construction, $\varphi(S) \cap \varphi(T) = \emptyset$ as well, so $\varphi(S)$ and $\varphi(T)$ are adjacent in $\mathrm{KG}(n',k')$. Thus, φ is a homomorphism.

Conversely, suppose there exists a homomorphism:

$$\psi : V(\mathrm{KG}(n,k)) \to V(\mathrm{KG}(n',k')).$$

Denote $\gamma := n/k = n'/k'$. So $\chi_v(\mathrm{KG}(n,k)) = \chi_v(\mathrm{KG}(n',k')) = \gamma$. Let $\mathbf{p}$ be the canonical vector coloring of $\mathrm{KG}(n,k)$, which was constructed at the end of Section 3.2. By the remark at the end of Section 3.2, we have that if $S, T \in V(\mathrm{KG}(n,k))$ with $h := |S \cap T|$, then:

$$\langle p_S, p_T \rangle = \frac{h}{k} \cdot \frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1}.$$

By Lemma 4.7, it follows that:

$$\left\{ \frac{h}{k} \cdot \frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} : h \in [k] \right\} \subset \left\{ \frac{h'}{k'} \cdot \frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} : h' \in [k] \right\}. \tag{4.1}$$

By (4.1), we have that for $h = 1$, there exists $h' \in [k']$ such that:

$$\frac{1}{k} \cdot \frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} = \frac{h'}{k'} \cdot \frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1}. \tag{4.2}$$

Note that (4.2) is equivalent to $k' = kh'$. Now as $n/k = n'/k'$, it follows that $n' = nh'$. So n' is an integer multiple of n, and r' is an integer multiple of r. $\square$

Using Lemma 4.7, the authors in [5] established an analogous necessary condition for the existence of homomorphisms between q-Kneser graphs. The proof is analogous to Theorem 4.8, replacing n, k, and h with their quantum analogues. It remains an open problem to characterize the existence of a homomorphism between q-Kneser graphs.

Theorem 4.9. *Let n, k, q, n', k', q' be integers satisfying $n \geq 2k + 1$, $n' \geq 2k' + 1$, and $[n]_q/[k]_q = [n']_{q'}/[k']_{q'}$. If there exists a homomorphism $\varphi : q$-KG$(n, k) \to q'$-KG(n', k'), then:*

$$\left\{ \frac{[h]_q}{[k]_q} : h \in [k] \right\} \subset \left\{ \frac{[h']_{q'}}{[k']_{q'}} : h' \in [k'] \right\}.$$

In particular, $[n']_{q'}$ and $[k']_{q'}$ are integer multiples of $[n]_q$ and $[k]_q$, respectively.

Proof. Let $\varphi : q\text{-KG}(n, k) \to q'\text{-KG}(n', k')$ be a homomorphism. Denote $\gamma := [n]_q/[k]_q = [n']_{q'}/[k']_{q'}$. By the remark at the end of Section 3.3, we note that $\chi_v(q\text{-KG}(n, k)) = \chi_v(q'\text{-KG}(n', k')) = \gamma$. By Theorem 3.20, q-KG(n, k) and q-KG(n', k') are uniquely vector colorable. So by Lemma 4.7, it follows that:

$$\left\{ \frac{[h]_q}{[k]_q} : h \in [k] \right\} \subset \left\{ \frac{[h']_{q'}}{[k']_{q'}} : h' \in [k'] \right\}. \tag{4.3}$$

Note that $[1]_q = 1$. So if $h = 1$, there exists $h' \in [k']$. such that:

$$\frac{1}{[k]_q} \cdot \frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} = \frac{[h']_{q'}}{[k']_{q'}} \cdot \frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1}. \tag{4.4}$$

Note that (4.4) is equivalent to: $[k']_{q'} = [k]_q[h']_{q'}$. As $[n]_q/[k]_q = [n']_{q'}/[k']_{q'}$, it follows that $[n']_{q'} = [n]_q[h']_{q'}$. $\square$

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